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ARTICLE I.

HINTS TO YOUNG STUDENTS.—No. VI.

30. I WOULD again remind the student that the object of Algebra is not alone that of enabling him to solve difficult questions, which are often, in themselves, of little value; but to instruct him in the language of symbols—a language which he will find to be the key to the higher branches of science, and, ultimately, to the profoundest mysteries of Nature. How little this is attended to in the usual course of study, may be seen in the little power which even advanced students generally have of comprehending or interpreting their symbolical results. The most simple kind of interpretation, that of transforming Algebraical into Arithmetical results should be constantly practised throughout the course. Of this kind a very good exercise is to take formulas, such as those in Equations (6) . . . . (11), which are true for all numbers, and *verify* them in particular cases. The proof that his work is correct will be in the indentity of the values he finds for the two members of the equation. Thus, in equation (9), if $a=\frac{1}{2}$, $b=\frac{1}{3}$,

$$(a+b)(a-b)=(\tfrac{1}{2}+\tfrac{1}{3})(\tfrac{1}{2}-\tfrac{1}{3})=\tfrac{5}{6}\cdot\tfrac{1}{6}=\tfrac{5}{36};$$
$$a^2-b^2=(\tfrac{1}{2})^2-(\tfrac{1}{3})^2=\tfrac{1}{4}-\tfrac{1}{9}=\tfrac{5}{36}.$$

He should continue his substitutions by putting, for a and b, such numbers as the following, in each of the equations:

1 and 0, 0 and -1, 100 and 1, 17 and 7, &c.;

$\frac{2}{3}$ and $\frac{1}{3}$, $\frac{5}{9}$ and $\frac{5}{6}$, $\frac{3}{4}$ and $\frac{7}{8}$, $\frac{4}{5}$ and $\frac{5}{4}$, &c.;

$1\frac{1}{3}$ and 1, $\dfrac{2\frac{1}{2}}{5}$ and -1, $3\frac{1}{8}$ and $-\dfrac{2\frac{1}{4}}{1\frac{1}{2}}$, -3 and $-2\frac{1}{3}$, &c.;

2.01 and ,1 . ,67 and ,7, -3, 1 and ,01, ,5 and ,05, &c.;

$\sqrt{2}$ and 1, $\sqrt{3}$ and $\sqrt{2}$, $\sqrt{\frac{1}{2}}$ and $\sqrt{\frac{1}{3}}$, $\sqrt[3]{5}$ and $\sqrt{3}$, &c.;

$$\sqrt{2}+1 \text{ and } \sqrt{2}-1,\ \sqrt{3}+\sqrt{2}, \text{ and } \sqrt{3}-\sqrt{2},\ \sqrt[3]{3}-\sqrt{2} \text{ and } \sqrt[3]{3}+\sqrt{5}, \text{ \&c.};$$

$$2^{\frac{1}{2}} \text{ and } 3^{\frac{1}{3}},\ \tfrac{1}{2}.2^{\frac{2}{3}} \text{ and } \tfrac{1}{3}.3^{\frac{1}{2}},\ \tfrac{1}{7}.(\tfrac{3}{4})^{\frac{3}{4}} \text{ and } \tfrac{5}{7}(\tfrac{4}{3})^{\frac{1}{4}}, \text{ \&c.};$$

$$c+d \text{ and } c-d,\ cx+d \text{ and } cx-d,\ cx+d \text{ and } ex+f, \text{ \&c.};$$

$$\frac{c}{d} \text{ and } \frac{c-d}{c+d},\ \frac{cx-dy}{cx+dy} \text{ and } \frac{c-d}{c+d},\ \frac{1+x}{x} \text{ and } \frac{x}{1-x}, \text{ \&c.};$$

$$a+\sqrt{b} \text{ and } b-\sqrt{a},\ \sqrt{a}+\sqrt{b} \text{ and } \sqrt{a}-\sqrt{b},\ \sqrt{\tfrac{1}{2}(n+1)} \text{ and } \sqrt{\tfrac{1}{2}(n-1)}, \text{ \&c.};$$

$$\frac{x}{\sqrt{x}+\sqrt{y}} \text{ and } \frac{y}{\sqrt{x}-\sqrt{y}},\ \sqrt[3]{\frac{x}{y}}, \text{ and } \sqrt[3]{\frac{y}{x}},\ \sqrt{1-y} \text{ and } y\sqrt{1-x}, \text{ \&c.};$$

&c., &c.

In these particular cases, it will be seen which member of the equation is the more easily calculated, and thus in any future case when a result is presented in one form, it can be seen, from the nature of the substitution, to what form it must be reduced so as to admit of the shortest calculation. The Algebraical substitutions will again form so many general formulas, which may be again submitted to verification, and which will often be worthy of special attention. Thus, for example, it will be seen that

$$\{\sqrt{\tfrac{1}{2}(n+1)}+\sqrt{\tfrac{1}{2}(n-1)}\}\{\sqrt{\tfrac{1}{2}(n+1)}-\sqrt{\tfrac{1}{2}(n-1)}\}=1:$$

then $\sqrt{\tfrac{1}{2}(n+1)}+\sqrt{\tfrac{1}{2}(n-1)}$ and $\sqrt{\tfrac{1}{2}(n+1)}-\sqrt{\tfrac{1}{2}(n-1)}$,

is one of the general forms for all numbers which are reciprocals of each other, and may be compared with similar forms, such as

$$\frac{p}{q} \text{ and } \frac{q}{p},$$

which possesses the same property; thus, if we put

$$\sqrt{\tfrac{1}{2}(n+1)}+\sqrt{\tfrac{1}{2}(n-1)}=\frac{p}{q},$$

then solve this equation for n, and substitute the result, we shall find

$$\sqrt{\tfrac{1}{2}(n+1)}-\sqrt{\tfrac{1}{2}(n-1)}=\frac{q}{p}.$$

It is familiarity with such properties, and facility in applying them, that enables the analyst to push his researches into fields of inquiry, which would otherwise be beyond his reach; since his symbols would become so complex that he could neither operate upon, nor interpret them. A single instance, in a simple case, will exhibit this. Equations of the form

$$\sqrt{x^2+ax+b}+\sqrt{x^2-ax+b}=c,$$

are frequently met with in Geometry. The usual process for freeing the equation of radicals, would be very complicated; but the analyst seizes the truth, which the form of the equation presents to him,

$$\{\sqrt{x^2+ax+b}+\sqrt{x^2-ax+b}\}\{\sqrt{x^2+ax+b}-\sqrt{x^2-ax+b}\}=2ax;$$

then $\sqrt{x^2+ax+b}-\sqrt{x^2-ax+b}=\frac{2ax}{c}$,

$$\sqrt{x^2+ax+b}=\frac{ax}{c}+\tfrac{1}{2}c,$$

$$x^2+ax+b=\frac{a^2x^2}{c^2}+ax+\tfrac{1}{4}c^2,$$

$$\left(1-\frac{a^2}{c^2}\right)x^2=\tfrac{1}{4}c^2-b,$$

$$x=\pm\frac{c}{2}\sqrt{\frac{c^2-4b}{c^2-a^2}}.$$

31. I have placed here a number of other identical formulas which are well adapted for verification and transformation, and they can be multiplied at pleasure.

$$(x+a)(x+b)=x^2+(a+b)x+ab,$$

$$(x+a)(x+b)(x+c)=x^3+(a+b+c)x^2+(ab+ac+bc)x+abc,$$

$$(ax-by)^2+(ay+bx)^2=(a^2+b^2)(x^2+y^2,)$$

$$(p^2-aq^2)^2+a(2pq)^2=(p^2+aq^2)^2,$$

$$(p^2-aq^2)^2+b(p^2-aq^2)(2pq+bq^2)+a(2pq+bq^2)=(p^2+bpq+aq^2)^2,$$

$$(p^3+apq^2)^2+a(pq^2+aq^3)^2=(p^2+aq^2)^3,$$

$$(p^4-6ap^2q^2+a^2q^4)^2+a(4p^3q-4apq^3)^2=(p^2+aq^2)^4,$$

$$(x+y)(xy-z^2)+(x+z)(xz-y^2)+(y+z)(yz-x^2)=0,$$

$$(x+y+z)^2+(x-y)^2+(x-z)^2+(y-z)^2=3(x^2+y^2+z^2),$$

$$\frac{n(n+1)(n+2)}{1.2.3}+\frac{n(n+1}{1.2}+n+1=\frac{(n+1)(n+2)(n+3)}{1.2.3},$$

$$x(x-1)(x-2)+3x(x-1)+x=x^3,$$

$$x(x-1)(x-2)(x-3)+6x(x-1)(x-2)+7x(x-1)+x=x^4,$$

$$x(x-1)+4xy+y(y-1)=(x+y)(x+y-1),$$

$$x(x-1)(x-2)+3x(x-1)y+3xy(y-1)+y(y-1)(y-2)$$
$$=(x+y)(x+y-1)(x+y-2),$$

$$x(x-1)+4x+2=(x+1)(x+2)$$

$$x(x-1)(x-2)+9x(x-1)+18x+6=(x+1)(x+2)(x+3),$$

$$\frac{b}{a+b}+\frac{a-b}{b}=\frac{a^2}{ab+b^2},$$

$$\frac{a-b}{a+b}+\frac{a+b}{a-b}=\frac{2a^2+2b^2}{a^2-b^2},$$

$$\frac{x^2-2x+1}{x^2-1}=\frac{x-1}{x+1},$$

$$\frac{x^4+x^2y^2+y^4}{x^3+y^3}=x+y-\frac{xy}{x+y},$$

$$\frac{x}{a}+\frac{bx}{a^2}+\frac{b^2x}{a^3}+\frac{b^3x}{a^3(a-b)}=\frac{x}{a-b},$$

$$\frac{x}{a}-\frac{bx}{a^2}+\frac{b^2x}{a^3}-\frac{b^3x}{a^3(a+b)}=\frac{x}{a+b},$$

$$\frac{a+x}{a-x}-\frac{a-x}{a+x}-\frac{4ax}{a^2-x^2}=0,$$

$$\frac{a(a^2-bc)+b(b^2-ac)+c(c^2-ab)}{a+b+c}=a^2-bc+b^2-ac+c^2-ab,$$

$$\frac{1}{(x-y)(x-z)}+\frac{1}{(y-x)(y-z)}+\frac{1}{(z-x)(z-y)}=0,$$

$$\frac{x}{(x-y)(x-z)}+\frac{y}{(y-x)(y-z)}+\frac{z}{(z-x)(z-y)}=0,$$

$$\frac{x^2}{(x-y)(x-z)}+\frac{y^2}{(y-x)(y-z)}+\frac{z^2}{(z-x)(z-y)}=1,$$

$$\frac{(a-y)(a-z)}{(x-y)(x-z)}+\frac{(a-x)(a-z)}{(y-x)(y-z)}+\frac{(a-x)(a-y)}{(z-x)(z-y)}=1,$$

$$x\,.\frac{(a-y)(a-z)}{(x-y)(x-z)}+y\,.\frac{(a-x)(a-z)}{(y-x)(y-z)}+z\,.\frac{(a-x)(a-y)}{(z-x)(z-y)}=a,$$

$$x^2.\frac{(a-y)(a-z)}{(x-y)(x-z)}+y^2.\frac{(a-x)(a-z)}{(y-x)(y-z)}+z^2.\frac{(a-x)(a-y)}{(z-x)(z-y)}=a^2,$$

$$(\sqrt{a}+\sqrt{b})(\sqrt{a}-\sqrt{b})=a-b,$$

$$\frac{x}{y-\sqrt{z}}=\frac{x(y+\sqrt{z})}{y^2-z},$$

$$\frac{x}{y+\sqrt{z}}=\frac{x(y-\sqrt{z})}{y^2-z},$$

$$\frac{x}{x-y}+\sqrt{\frac{x^2}{(x-y)^2}-\frac{x}{x-y}}=\frac{\sqrt{x}}{\sqrt{x}-\sqrt{y}},$$

$$\frac{x}{x-y}-\sqrt{\frac{x^2}{(x-y)^2}-\frac{x}{x-y}}=\frac{\sqrt{x}}{\sqrt{x}+\sqrt{y}},$$

$$\frac{y}{\sqrt{x^2+zy}-x}=\frac{x+\sqrt{x^2+zy}}{z}$$

$$\sqrt{\frac{x+a}{x}}+2\sqrt{\frac{a}{a+x}}=\frac{(\sqrt{a}+\sqrt{x})^2.}{\sqrt{ax+x^2}}$$

$$\sqrt{xy+2x\sqrt{xy-x^2}}+\sqrt{xy-2x\sqrt{xy-y^2}}=2x,$$

$$\sqrt{x}+\sqrt{y}=\sqrt{\tfrac{1}{2}x+\tfrac{1}{2}\sqrt{x^2-y}}+\sqrt{\tfrac{1}{2}x-\tfrac{1}{2}\sqrt{x^2-y}},$$

$$\sqrt{x}-\sqrt{y}=\sqrt{\tfrac{1}{2}x+\tfrac{1}{2}\sqrt{x^2-y}}-\sqrt{\tfrac{1}{2}x-\tfrac{1}{2}\sqrt{x^2-y}},$$

$$\sqrt{x+y\sqrt{-1}}=\sqrt{\tfrac{1}{2}\sqrt{x^2+y^2}+\tfrac{1}{2}x}+\sqrt{-1}.\sqrt{\tfrac{1}{2}\sqrt{x^2+y^2}-\tfrac{1}{2}x},$$

$$\sqrt{x-y\sqrt{-1}}=\sqrt{\tfrac{1}{2}\sqrt{x^2+y^2}+\tfrac{1}{2}y}-\sqrt{-1}.\sqrt{\tfrac{1}{2}\sqrt{x^2+y^2}-\tfrac{1}{2}x},$$

$$\sqrt{x+\sqrt{y}}+\sqrt{x-\sqrt{y}}=\sqrt{2x+2\sqrt{x^2-y}},$$

$$\sqrt{x+\sqrt{y}}-\sqrt{x-\sqrt{y}}=\sqrt{2x-2\sqrt{x^2-y}},$$

$$\sqrt{x+y\sqrt{-1}}+\sqrt{x-y\sqrt{-1}}=\sqrt{2\sqrt{x^2+y^2}+2x},$$

$$\sqrt{x+y\sqrt{-1}}-\sqrt{x-y\sqrt{-1}}=\sqrt{-1}.\sqrt{2\sqrt{x^2+y^2}-2x},$$

$$\frac{\sqrt{\sqrt{x}+\sqrt{y}}+\sqrt{\sqrt{x}-\sqrt{y}}}{\sqrt{\sqrt{x}+\sqrt{y}}-\sqrt{\sqrt{x}-\sqrt{y}}}=\frac{x+\sqrt{x^2-y}}{\sqrt{y}},$$
$$\frac{\sqrt{\sqrt{x}+\sqrt{y}}-\sqrt{\sqrt{x}-\sqrt{y}}}{\sqrt{\sqrt{x}+\sqrt{y}}+\sqrt{\sqrt{x}+\sqrt{y}}}=\frac{x-\sqrt{x^2-y}}{\sqrt{y}};\ \&c.$$

These should first be verified generally, that is, the two members of each equation should be so operated upon as to render them identical; and afterwards for particular values, and among the numerical values of the letters, 0 and 1 must be often used; thus in the equation

$$\sqrt{x-y\sqrt{-1}}=\sqrt{\tfrac{1}{2}\sqrt{x^2+y^2}+\tfrac{1}{2}x}-\sqrt{-1}.\sqrt{\tfrac{1}{2}\sqrt{x^2+y^2}-\tfrac{1}{2}x},$$

if $x=0, y=2$, $\quad\sqrt{-2\sqrt{-1}}=1-\sqrt{-1}$,

which may be verified by squaring both members, when they will be found to be identical. A very useful exercise consists in taking each member of such equations, and deducing the other one from it by successive transformations, thus:

$$\begin{aligned} x(x-1)(x-2)+3x(x-1)+x &= x\{(x-1)(x-2)+3(x-1)+1\} \\ &= x\{(x-1)(x-2+3)+1\} \\ &= x\{(x-1)(x+1)+1\} \\ &= x(x^2-1+1) \\ &= x^3; \end{aligned}$$

so,
$$\begin{aligned} x^3 &= x.x.x \\ &= x(x-1+1)(x-2+2) \\ &= x\{(x-1)(x-2)+2(x-1)+x-1-1+2\} \\ &= x\{(x-1)(x-2)+3(x-1)+1\} \\ &= x(x-1)(x-2)+3x(x-1)+x. \end{aligned}$$

32. In the following equations, n is supposed to be a positive integer, although many of them are true for any values of n whatever. They should be verified for particular values of n, and several of the equations given above are among the particular cases of these,

$$(x+y)^n=x^n+\frac{n}{1}.x^{n-1}y+\frac{n.(n-1)}{1.2}.x^{n-2}y^2+\ldots\ldots\frac{n}{1}.xy^{n-1}+y^n.$$

$$2^n=1+\frac{n}{1}+\frac{n(n-1)}{1.2}+\frac{n(n-1)(n-2)}{1.2.3}+\ldots\frac{n}{1}+1.$$

$$0=1-\frac{n}{1}+\frac{n(n-1)}{1.2}-\frac{n(n-1)(n-2)}{1.2.3}+\&c.,$$

$$n^n-\frac{n}{1}(n-1)^n+\frac{n(n-1)}{1.2}(n-2)^n-\&c.\ldots=1.2.3\ldots n,$$

$$(n-1)^{n-1}-\frac{n}{1}(n-2)^{n-1}+\frac{n(n-1)}{1.2}(n-3)^{n-1}-\&c.\ldots=1,$$

$$(n-2)^{n-1}-\frac{n}{1}(n-3)^{n-1}+\frac{n(n-1)}{1.2}(n-4)^{n-1}-\&c.\ldots=2^{n-1}-n,$$

$$(n-3)^{n-1}-\frac{n}{1}(n-4)^{n-1}+\frac{n(n-1)}{1.2}(n-5)^{n-1}-\&c.\ldots=3^{n-1}-2^{n-1}n+\frac{n(n-1)}{1.2},$$

$$(n+1)(n+2)(n+3)\ldots(n+n)=2^n\times1.3.5.7\ldots(2n-1)$$

$$\frac{(m+1)(m+2)(m+3)\ldots(m+n)}{1.2.3\ldots n}=\frac{(n+1)(n+2)(n+3)\ldots(n+m)}{1.2.3\ldots m}$$

$$\frac{(n+1)(n+2)\ldots\ldots(n+n)}{1.2.3\ldots\ldots n}=1+\left(\frac{n}{1}\right)^2+\left(\frac{n.\overline{n-1}}{1.2}\right)^2+\left(\frac{n.\overline{n-1}.\overline{n-2}}{1.2.3}\right)^2$$
$$+\ldots\ldots\left(\frac{n}{1}\right)^2+1,$$

$$(x+1)(x+2)\ldots\ldots(x+n)=x(x-1)\ldots\ldots(x-n+1)+\frac{n^2}{1}.x(x-1)\ldots\ldots(x-n+2)$$
$$+\frac{n^2(n-1)^2}{1.2}.x(x-1)\ldots\ldots(x-n+3)+\&c.\ldots.$$
$$+\frac{n^2(n-1)^2\ldots\ldots2^2}{1.2.3\ldots\ldots\overline{n-1}}.x+\frac{n^2(n-1)^2(n-2)^2\ldots\ldots1^2}{1.2.3\ldots\ldots n},$$

$$(x+y)(x+y-1)..(x+y-n+1)=x(x-1)(x-2)\ldots\ldots(x-n+1)$$
$$+\frac{n}{1}.x(x-1)\ldots\ldots(x-n+2).y$$
$$+\frac{n(n-1)}{1.2}.x(x-1)..(x-n+3).y(y-1)+\&c..$$
$$+\frac{n}{1}.x,y(y+1)\ldots(y-n+2)$$
$$+y(y-1)(y-2)\ldots(y-n+1),$$

$$(x+y)(x+y+1)..(x+y+n-1)=x(x+1)(x+2)\ldots(x+n-1)$$
$$+\frac{n}{1}.x(x+1)\ldots(x+n-2).y$$
$$+\frac{n(n-1)}{1.2}.x(x+1)..(x+n-3).y(y+1)+\&c..$$
$$+\frac{n}{1}.xy.(y+1)\ldots(y+n-2)$$
$$+y(y+1)(y+2)\ldots(y+n-1),$$

$$(a_1+a_2+\ldots a_n)^2=a_1^{\,2}+a_2^{\,2}+\ldots+a_n^{\,2}+2a_1a_2+2a_1a_3+\ldots+2a_2a_3+\ldots,$$

$$(a_1+a_2+\ldots a_n)^2+(a_1-a_2)^2+(a_1-a_3)^2+\ldots+(a_2-a_3)^2+(a_2-a_4)^2+\ldots,$$
$$=n(a_1^{\,2}+a_2^{\,2}+\ldots\ldots a^2_n),$$

$$(a_1x_1+a_2x_2+\ldots a_nx_n)^2+(a_1x_2-a_2x_1)^2+(a_1x_3-a_3x_1)^2+\ldots$$
$$+(a_2x_3-a_3x_2)^2+(a_2x_4-a_4x_2)^2+\ldots$$
$$+\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$=(a_1^{\,2}+a_2^{\,2}+\ldots\ldots a_n^{\,2})(x_1^{\,2}+x_2^{\,2}+\ldots\ldots x_n^{\,2}),$$

$$0=\frac{1}{(x_1-x_2)(x_1-x_3)\ldots\ldots(x_1-x_n)}+\frac{1}{(x_2-x_1)(x_2-x_3)\ldots\ldots(x_2-x_n)}$$
$$+\&c.\ldots\ldots\ldots\ldots\ldots\ldots+\frac{1}{(x_n-x_1)(x_n-x_2)\ldots\ldots(x_n-x_{n-1})},$$

$$1=\frac{x_1^{\,n-1}}{(x_1-x_2)(x_1-x_3)\ldots\ldots(x_1-x_n)}+\frac{x_2^{\,n-1}}{(x_2-x_1)(x_2-x_3)\ldots\ldots(x_2-x_n)}$$
$$+\&c.\ldots\ldots\ldots\ldots\ldots\ldots+\frac{x_n^{\,n-1}}{(x_n-x_1)(x_n-x_2)\ldots\ldots(x_n-x_{n-1})}$$

$$1=\frac{(a-x_2)(a-x_3)\ldots\ldots(a-x_n)}{(x_1-x_2)(x_1-x_3)\ldots\ldots(x_1-x_n)}+\frac{(a-x_1)(a-x_3)\ldots\ldots(a-x_n)}{(x_2-x_1)(x_2-x_3)\ldots\ldots(x_2-x_n)}$$
$$+\&c.\ldots\ldots\ldots\ldots\ldots\ldots+\frac{(a-x_1)(a-x_2)\ldots\ldots(a-x_{n-1})}{(x_n-x_1)(x_n-x_2)\ldots\ldots(x_n-x_{n-1})},$$

$$0=n^i-\frac{n}{1}(n-1)^i+\frac{n(n-1)}{1\,.\,2}(n-2)^i-\frac{n(n-1)(n-2)}{1\,.\,2\,.\,3}(n-3)^i+\&c.\dots$$

$$0=\frac{x_1^{\,i-1}}{(x_1-x_2)(x_1-x_3)\dots(x_1-x_n)}+\frac{x_2^{\,i-1}}{(x_2-x_1)(x_2-x_3)\dots(x_2-x_n)}$$
$$+\ \&c.\ \dots\dots\dots\dots\dots+\frac{x_n^{\,i-1}}{(x_n-x_1)(x_n-x_2)\dots(x_n-x_{n-1})},$$

$$a=x_1^{\,i}\frac{(a-x_2)(a-x_3)\dots(a-x_n)}{(x_1-x_2)(x_1-x_3)\dots(x_1-x_n)}+x_2^{\,i}\frac{(a-x_1)(a-x_3)\dots(a-x_n)}{(x_2-x_1)(x_2-x_3)\dots(x_2-x_n)}$$
$$+\ \&c.\ \dots\dots\dots\dots\dots+x_n^{\,i}\frac{(a-x_1)(a-x_2)\dots(a-x_{n-1})}{(x_n-x_1)(x_n-x_2)\dots(x_n-x_{n-1})}.$$

The last three equations are true only so long as i is an integer less than n; x_1, x_2, x_3, &c. x_n, mean so many different numbers, and this notation is often found more convenient than that of using different letters, since the symbols may thus be made to indicate the rank of the numbers. In the advanced state of Algebra, it is essential that the student should make himself familiar with this species of notation. Exercises like the following will enable him to do so.

33. If the development of any power of a binomial be put in the form

$$(1+x)^m=A_0+A_1x+A_2x^2+\dots+A_nx^n+\dots,$$

so that A_n indicates the co-efficient of the n^{th} power of x, as, for instance, A_2 is the co-efficient of x^2, and A_{10} that of x^{10}; then it is found that

$$A_0=1,\ A_1=\frac{m}{1},\ A_2=\frac{m(m-1)}{1\,.\,2},\ A_3=\frac{m(m-1)(m-2}{1\,.\,2\,.\,3},\ \&c.$$

Now in order to deduce the expression for A_n, or the co-efficient of *any* power of n, in the development, we remark that the value of A_1 is a fraction having one factor in each member, that A_2 is a fraction having two factors iu each member, that A_3 is a fractiom having three factors in each member, &c. and we conclude that A_n is a fraction having n factors in each member. We next remark that the factors in the numerator successively decrease by an unit, the first factor being m, and therefore the last in the expression for A_n must be $m-(n-1)=m-n+1$; and the factors in the denominator successively increase by an unit, the first factor being 1, and therefore the last, in the expression for A_n, must be $1+(n-1)=n$; hence

$$A_n=\frac{m(m-1)(m-2)\dots(m-n+1)}{1\,.\,2\,.\,3\dots n},$$

the numerator expressing the continued product of all numbers, differing by unity, from m to $m-n+1$ inclusive, and the denominator these from 1 to n inclusive; and if in this result we write, instead of n, the successive values 1, 2, 3, 4, &c. we shall get the particular values for A_1, A_2, &c.

But the analyst endeavors to express the law of continuity, among such a series of numbers, by an Algebraic relation among any two or more consecutive ones of the series. Thus, in the case of the binomial co-efficients, it is seen that

$$\frac{A^1}{A}=\frac{m}{1},\ \frac{A_2}{A_1}=\frac{m-1}{2},\ \frac{A_3}{A_2}=\frac{m-2}{3},\ \frac{A_4}{A_3}=\frac{m-3}{4},\ \&c.,$$

and it is concluded that

$$\frac{A_n}{A_{n-1}} = \frac{m-n+1}{n}, \text{ or } A_n = \frac{m-n+1}{n} . A_{n-1},$$

which, together with the fact that $A_0 = 1$, makes known the whole of the co-efficients; and therefore these conclusions should be first aimed at, and the particular ones deduced from them, by the successive substitution of 1, 2, 3, &c., for n; thus, since $A_0 = 1$,

$$A_1 = \frac{m-1+1}{1} . A_0 = \frac{m}{1},$$

$$A_2 = \frac{m-2+1}{2} . A_1 = \frac{m}{1} . \frac{m-1}{2},$$

$$A_3 = \frac{m-3+1}{3} . A_2 = \frac{m}{1} . \frac{m-1}{2} . \frac{m-2}{3},$$

&c.

The direct mode of investigating an expression for A_n, in terms of n and given things, from such relations as these, belongs to a higher branch of analysis; but in simple cases it may be deduced by analogy in the way directed above, as well as in other ways, of which this is one:—by successive substitution,

$$A_0 = 1,$$

$$A_1 = \frac{m}{1} . A_0,$$

$$A_2 = \frac{m-1}{2} . A_2,$$

$$A_3 = \frac{m-2}{3} . A_2,$$

$$A_4 = \frac{m-3}{4} . A_4,$$

$$\begin{matrix} . & . \\ . & . \\ . & . \end{matrix}$$

$$A_n = \frac{m-n+1}{n} . A_{n-1};$$

the continued product of the two members of these equations, gives

$$A_0 A_1 A_2 A_3 \ldots\ldots A_{n-1} A_n = \frac{m}{1} . \frac{m-1}{2} . \frac{m-2}{3} \ldots\ldots \frac{m-n+1}{n} . A_0 A_1 A_2 \ldots\ldots A_{n-1},$$

and dividing both members by $A_0 A_1 A_2 \ldots\ldots A_{n-1}$,

$$A_n = \frac{m}{1} . \frac{m-1}{2} . \frac{m-2}{3} \ldots\ldots \frac{m-n+1}{n}.$$

Many remarkable properties of these quantities may be deduced from these relations among them; thus, 1°. if $S . A_m$ represent their sum, by making $x = 1$,

$$(1+1)^m = A_0 + A_1 + A_2 + \ldots\ldots + A_m,$$

or $$S . A_m = 2^m.$$

2°. If m be an odd integer, or $m=2m'+1$, by making $n=m'+1$,

$$A_{m'+1}=\frac{2m'+1-m'-1+1}{m'+1}A_{m'}=A^{m'},$$

or the co-efficients of the two middle terms of such a development are equal.

3°. If m be any integer, the co-efficient standing the n^{th} from the last or A_m, may be represented by A_{m-n}, and writing $m-n$ instead of n, in the expression for A_n,

$$A_{m-n}=\frac{m}{1}.\frac{m-1}{2}.\frac{m-2}{3}\dots\frac{m+1}{m-n}$$

$$=\frac{m}{1}.\frac{m-1}{2}.\frac{m-2}{3}\dots\frac{m-n+1}{n}\times\frac{m-n}{n+1}.\frac{m-n-1}{n+2}\dots\frac{n+1}{m-n}$$

$$=\frac{m}{1}.\frac{m-1}{2}.\frac{m-2}{3}\dots\frac{m-n+1}{n}$$

$$=A_n,$$

or the co-efficients of the terms, equally distant in rank from the extreme terms, are the same. The student will do well to investigate in this manner, the following examples, and others of the same kind,

$$A_n-A_{n-1}=d,\qquad \text{and } A_0=a;$$

$$A_n=rA_{n-1},\qquad \text{and } A_0=a;$$

$$nA_n=(n+m)A_{n-1},\ \text{and } A_0=1=\frac{1.2.3\dots m}{1.2.3\dots m};$$

$$A_n-A_{n-1}=(m-2)n+1,\ \text{and } A_0=1=\frac{1.2}{1.2};$$

$$A_n-2A_{n-2}+A_{n-2}=2d^2,\qquad \text{and } A_0=a^2,\ A_1=(a+d)^2$$

$$(n+1)A_n+nA_{n-1}=0,\ A_0=a;$$

$$n(n+1)A_n=aA_{n-1},\ A_0=1;$$

$$n(n-1)A_n=aA_{n-2},\ A_0=1,\ A_1=0;$$

$$(2n+a)A_n=(2n+a-1)A_{n-1},\ A_0=1;$$

$$2nA_n=(2n-3)(2n-1)A_{n-2},A_0=1,\ A_1=\frac{1}{2^2}$$

$$2n(2n+a-1)A_n=bA_{n-1},\ A_0=1.$$

It will be seen that the first equation expresses the property of a series of numbers A_0, A_1, &c., in which any two consecutive terms differ by a constant quantity, d, and these numbers will therefore form a progression by differences; similarly, the numbers found from the second equation will be a progression by quotients; those from the third, the figurate numbers, of the order m; those from the fourth, the polygonal numbers, of the order m, &c.

34. In solving equations, different processes should be used, and the results should be verified, by substitution; and this will often be a more useful exercise than the solution of the equation. In the equation

$$x-\frac{ax}{a+b}+c=\frac{ac}{a-b}-\frac{b^2x}{a^2-b^2};$$

the fractions could be made to disappear and the terms collected in the usual manner; or, since,

$$x-\frac{ax}{a+b}=x\left(1-\frac{a}{a+b}\right)=x.\frac{b}{a+b},$$

$$\text{and}\quad \frac{ac}{a-b}-c=c\left(\frac{a}{a-b}-1\right)=c.\frac{b}{a-b},$$

the equation becomes, after dividing by b,

$$\frac{x}{a+b}=\frac{c}{a-b}-\frac{bx}{a^2-b^2},$$

$$\frac{x}{a+b}+\frac{bx}{a^2-b^2}=\frac{c}{a-b},$$

$$\frac{ax}{a^2-b^2}=\frac{c}{a-b},$$

$$x=\frac{c(a+b)}{a}=c\left(1+\frac{b}{a}\right).$$

To verify this result,

$$x-\frac{ax}{a+b}+c=c\left(1+\frac{b}{a}\right)-c+c=c\left(1+\frac{b}{a}\right);$$

and

$$\frac{ac}{a-b}-\frac{b^2x}{a^2-b^2}=\frac{ac}{a-b}-\frac{b^2}{a^2-b^2}.\frac{c(a+b)}{a}$$

$$=\frac{ac}{a-b}-\frac{b^2c}{a(a-b)}$$

$$=\frac{(a^2-b^2)c}{a(a-b)}=\frac{(a+b)c}{a}=c\left(1+\frac{b}{a}\right).$$

The equation should also be solved by the most direct process, and the root afterwards reduced to its most simple form; thus, by collecting the terms, as if the co-efficients were simple,

$$x\left(1-\frac{a}{a+b}+\frac{b^2}{a^2-b^2}\right)=c\left(\frac{a}{a-b}-1\right),$$

$$x=c.\frac{\frac{a}{a-b}-1}{1-\frac{a}{a+b}+\frac{b^2}{a^2-b^2}}$$

$$=c.\frac{a^2-b^2}{a^2-b^2}.\frac{\frac{a}{a-b}-1}{1-\frac{a}{a+b}+\frac{b^2}{a^2-b^2}}$$

$$=c.\frac{a(a+b)-(a^2-b^2)}{a^2-b^2-a(a-b)+b^2}$$

$$=c.\frac{a^2+ab-a^2+b^2}{a^2-b^2-a^2+ab+b^2}$$

$$=c.\frac{ab+b^2}{ab}$$

$$=c\left(1+\frac{b}{a}\right).$$

ARTICLE II.

SOLUTIONS TO THE QUESTIONS PROPOSED IN NUMBER VI.

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(31). QUESTION I. *By* ——.

Given $a =$,280796, $b = 1{,}528307$, $a = 3$, $d =$,087648, $e =$,002879; to calculate the numerical value of the expression

$$x = \sqrt{(ab + c)}\,\frac{d}{e},$$

true to five places of decimals; and exhibit the work, without using logarithms.

SOLUTION. *By Mr. E. H. Delafield, St. Paul's College.*

$$\begin{array}{r}
,2807960 \\
\underline{703825,1} \\
2807960 \\
1403980 \\
56159 \\
22463 \\
842 \\
\underline{20} \\
ab = \ ,4291424 \\
\underline{c = 3,}
\end{array}$$

$$\begin{array}{r|l}
,087648 & ,002879 \\
8637 & 30{,}443904 = \dfrac{d}{e} \\
12780 & \\
11516 & \\
12640 & \\
11516 & \\
11240 & \\
8637 & \\
2603 & \\
2591 & \\
12 & \\
11 &
\end{array}$$

$ab + c = 3{,}4291424 \,|\, 1{,}85179437 = \sqrt{ab + c}$.

$$\begin{array}{r|l}
 & 1 \\
28 & 242 \\
 & 224 \\
365 & 1891 \\
 & 1825 \\
3701 & 6642 \\
 & 3701 \\
37027 & 294140 \\
 & 259189 \\
37034 & 34951 \\
\cdots & 33331 \\
 & 1620 \\
 & 1481 \\
 & 139 \\
 & 111 \\
 & 28 \\
 & 26
\end{array}$$

$$\begin{array}{r}
30{,}443904 \\
\underline{73497158{,}1} \\
30443904 \\
24355123 \\
1522195 \\
30444 \\
21311 \\
2740 \\
122 \\
9 \\
2
\end{array}$$

$$\frac{d}{e}\sqrt{ab+c} = 56{,}375850 = x.$$

(32). QUESTION II. *By* ——.

Express the number 1006005 in a system of notation whose scale of relation is 6.

SOLUTION. *By Mr. W. B. Benedict, Upperville, Va.*

To transform a number N, from the denary to any other system of notation, in which the scale of relation is R, it is only necessary to determine the digits or co-efficients, a, b, c, ... f, g, h, in the general expression

$$N = aR^n + bR^{n-1} + cR^{n-2} + \dots + fR^2 + gR + h.$$

If we divide this equation by R, and denote by N$'$ the quotient of N by R, and by r the remainder, we get

$$N' + \frac{r}{R} = aR^{n-1} + bR^{n-2} + cR^{n-3} + \dots + fR + g + \frac{h}{R};$$

where the fractions and whole numbers in the two members must be identical, then

$$h = r,$$

$$N' = aR^{n-1} + bR^{n-2} + cR^{n-3} + \dots + fR + g.$$

In the same manner g is the remainder after dividing N$'$ by R, f the remainder after dividing the last quotient by R, &c. In the present case, R = 6, N = 1006005; hence

$$\begin{array}{r|l}
6 & 1006005 \\
6 & 167667 + \frac{3}{6}, \text{ or } h = 3, \\
6 & 27944 + \frac{3}{6}, \text{ or } g = 3, \\
6 & 4657 + \frac{2}{6}, \text{ or } f = 2, \\
6 & 776 + \frac{1}{6}, \text{ or } e = 1, \\
6 & 129 + \frac{2}{6}, \text{ or } d = 2, \\
6 & 21 + \frac{3}{6}, \text{ or } c = 3, \\
6 & 3 + \frac{3}{6}, \text{ or } b = 3, \\
 & 0 + \frac{3}{6}, \text{ or } a = 3;
\end{array}$$

and the number in the senary scale, is

33321233.

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(33). QUESTION III. *By* ——.

Given, to find x and y, the two equations

$$x + \frac{x^2}{y} + y = a,$$

$$x^2 + \frac{x^4}{y^2} + y^2 = b.$$

FIRST SOLUTION. *By Mr. D. D. Hughes, Syracuse Academy.*

Squaring the first equation, we get

$$x^2+\frac{x^4}{y^2}+y^2+\frac{2x^3}{y}+2xy+2x^2=a^2\,;$$

Subtract the second, $$\frac{2x^3}{y}+2xy+2x^2=a^2-b\,;$$

Divide this by the first equation, member by member,

$$2x=\frac{a^2-b}{a},\text{ or }x=\frac{a^2-b}{2a}.$$

By clearing the first equation of the fractions, and reducing,

$$y^2-(a-x)y=-x^2,$$

and $$y=\tfrac{1}{2}(a-x)\pm\tfrac{1}{2}\sqrt{(a-x)^2-4x^2}$$

$$=\frac{a^2+b}{4a}\pm\tfrac{1}{2}\sqrt{\frac{(a^2+b)^2}{4a^2}-\frac{(a^2-b)^2}{a^2}}.$$

SECOND SOLUTION. *By Mr. J. V. Campbell, St. Paul's College.*

Divide the second equation by the first, then

$$-x+\frac{x^2}{y}+y=\frac{b}{a}\;.\;.\;.\;.\;.\;.\;.\;.\;(3),$$

subtract (3) from the first, then

$$2x=a-\frac{b}{a},$$

or $$x=\frac{a^2-b}{2a},$$

and $$x^2=\frac{(a^2-b)^2}{4a^2}\;.\;.\;.\;.\;.\;.\;.\;.\;.\;(4).$$

Add (4) to the second equation, then

$$\frac{x^4}{y^2}+2x^2+y^2=b+\frac{(a^2-b)^2}{4a^2}=\frac{(a^2+b)^2}{4a^2},$$

and $$y+\frac{x^2}{y}=\frac{a^2+b}{2a}\;.\;.\;.\;.\;.\;.\;.\;.\;.\;(5).$$

Multiply (4) by (3), and subtract from the second equation,

$$\frac{x^4}{y^2}-2x^2+y^2=b-\frac{3(a^2-b)^2}{4a^2}=\frac{4a^2b-3(a^2-b)^2}{4a^2},$$

and $$y-\frac{x^2}{y}=\pm\frac{1}{2a}\sqrt{4a^2b-3(a^2-b)^2}\;.\;.\;.\;.\;(6),$$

Add (5) and (6) $$y=\frac{1}{4a}\left(a^2+b\pm\sqrt{4a^2b-3(a^2-b)^2}\right).$$

3
(24). QUESTION. *By the Editor.* 3

Given that

$$2(a+b)^2+ab=(2a+b)(a+2b).$$

It is required to divide the number

$$2x+y$$

into two factors.

SOLUTION. *By Mr. J. Blickensderfer, jun., Canal Dover, Ohio.*

If we make $(a+b)^2 = x$, and $ab = y$;
so that $(a-b)^2 = x - 4y$, and therefore

$$a + b = \sqrt{x}, \text{ and } a - b = \sqrt{x - 4y};$$

or $$a = \tfrac{1}{2}\sqrt{x} + \tfrac{1}{2}\sqrt{x-4y}, \text{ and } b = \tfrac{1}{2}\sqrt{x} - \tfrac{1}{2}\sqrt{x-4y};$$

the two factors will be

$$2a + b = \tfrac{3}{2}\sqrt{x} + \tfrac{1}{2}\sqrt{x-4y}, \text{ and } a + 2b = \tfrac{3}{2}\sqrt{x} - \tfrac{1}{2}\sqrt{x-4y};$$

therefore $$2x + y = \tfrac{1}{2}(3\sqrt{x} + \sqrt{x-4y}) \times \tfrac{1}{2}(3\sqrt{x} - \sqrt{x-4y}).$$

—— *Cor.* If we put $2x + y = n$, so that $y = n - 2x$, we have

$$n = \tfrac{1}{2}(3\sqrt{x} + \sqrt{9x-4n}) \times \tfrac{1}{2}(3\sqrt{x} - \sqrt{9x-4n});$$

where x may be any number whatever, and this enables us to divide any number n, an infinite number of ways, into two irrational factors; perhaps a neater form for them is

$$n = (x + \sqrt{x^2 - n})(x - \sqrt{x^2 - n}).$$

(35.) QUESTION V. *From Peirce's Algebra.*

A, B, C, D, E play together on this condition, that he who loses shall give to all the rest as much as they already have. First A loses, then B, then C, then D, and at last also E. All lose in turn, and yet at the end of the fifth game they all have the same sum, viz., each $32. How much had each when they began to play?

SOLUTION. *By Omicron, jun., Chapel Hill, N. C.*

Let the original sum presented by A, B, C, D, E, be respectively represented by u, v, x, y, z; then, as they will have the same sum, in all, at first as at last,

$$u + v + x + y + z = 5 \times 32 = 160.$$

Now, since the person who loses pays to each of the rest as much as they already have, he must pay them all $160 — the money he has himself, and when his losses are paid he will have left twice what he had before—160, while each winner's money is doubled; hence at the end of the several successive games the money each man has may be represented thus:

	A has	B has	C has	D has	E has
1st game,	$2u - 160$	$2v$	$2x$	$2y$	$2z$,
2nd do.	$4u - 320$	$4v - 160$	$4x$	$4y$	$4z$,
3rd do.	$8u - 640$	$8v - 320$	$8x - 160$	$8y$	$8z$,
4th do.	$16u - 1280$	$16v - 640$	$16x - 320$	$16y - 160$	$16z$,
5th do.	$32u - 2560$	$32v - 1280$	$32x - 640$	$32y - 320$	$32z - 160$;

Hence,
$$\begin{aligned}
32u - 2560 &= 32, &\text{and } u &= 80 + 1 = 81,\\
32v - 1280 &= 32, & v &= 40 + 1 = 41,\\
32x - 640 &= 32, & x &= 20 + 1 = 21,\\
32y - 320 &= 32, & y &= 10 + 1 = 11,\\
32z - 160 &= 32, & z &= 5 + 1 = 6.
\end{aligned}$$

—— Omicron, deduces from this a general solution, which will be reserved for Question (100) of the Senior Department. Prof. Peirce states that this question is taken from a much older book.

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(36). QUESTION VI. *By Mr. Geo. W. Coaklay, Peekskill Academy, N. Y.*

Find what relation must exist among the co-efficients of the equation

$$x^4 + \mathrm{A}x^3 + \mathrm{B}x^2 + \mathrm{C}x + \mathrm{D} = 0,$$

so that it may be put in either of the forms

$$(x^2 + ax)^2 + b(x^2 + ax) + c = 0,$$

or

$$(x^2 + a'x + b')^2 + c' = 0.$$

SOLUTION. *By Mr. B. Birdsall, New-Hartford, N. Y.*

By expanding the second equation, it becomes

$$x^4 + 2ax^3 + (a^2 + b)x^2 + abx + c = 0;$$

and if the first equation can be put in this form, it must be the case that

$$2a = \mathrm{A},\ a^2 + b = \mathrm{B},\ ab = \mathrm{C},\ c = \mathrm{D};$$

then $a = \frac{1}{2}\mathrm{A}$, $\quad b = \mathrm{B} - a^2 = \mathrm{B} - \frac{1}{4}\mathrm{A}^2$,

and, substituting these in the equation $ab = \mathrm{C}$,

$$\mathrm{AB} - \tfrac{1}{4}\mathrm{A}^3 = 2\mathrm{C};$$

which is the relation required. Similarly the third equation gives

$$x^4 + 2a'x^3 + (a'^2 + 2b')x^2 + 2a'b'x + b'^2 + c' = 0;$$

and if the first equation can be put in this form,

$$2a' = \mathrm{A},\ a'^2 + 2b' = \mathrm{B},\ 2a'b' = \mathrm{C},\ b'^2 + c' = \mathrm{D};$$

then $a' = \frac{1}{2}\mathrm{A}$, $b' = \frac{1}{2}\mathrm{B} - \frac{1}{2}a'^2 = \frac{1}{2}\mathrm{B} - \frac{1}{8}\mathrm{A}^2$,

and the third becomes

$$\mathrm{AB} - \tfrac{1}{4}\mathrm{A}^3 = 2\mathrm{C},$$

which is the same relation as before; and therefore if an equation of the fourth degree can be put in one of these forms, it can be put in the other one, and the roots in either form can be found by the usual process for equations of the second degree.

—— Mr. Coaklay, the proposer, solved question (24) by this method, thus: In that question, $\mathrm{A} = -3$, $\mathrm{B} = -8\frac{3}{4}$, $\mathrm{C} = 16\frac{1}{2}$, $\mathrm{D} = -2075\frac{3}{4}$, which have the necessary relation; therefore, $a' = -\frac{3}{2}$, $b' = -\frac{11}{2}$, $c' = \mathrm{D} - b'^2 = -2106$, and the equation may be put in the form

$$(x^2 - \tfrac{3}{2}x - \tfrac{11}{2})^2 - 2106 = 0, \text{ \&c.}$$

(37.) QUESTION VII. *By β.*

Prove that, if θ be any angle,

$$\tan^2\theta - \tan^2\tfrac{1}{2}\theta = \frac{8\sin^2\frac{1}{2}\theta\cos\frac{3}{2}\theta}{\cos^2\theta}.$$

FIRST SOLUTION. *By Mr. Geo. W. Coaklay.*

$$\tan^2\theta - \tan^2\tfrac{1}{2}\theta = (\tan\theta + \tan\tfrac{1}{2}\theta)(\tan\theta - \tan\tfrac{1}{2}\theta).$$

But, $\quad \tan\theta + \tan\frac{1}{2}\theta = \dfrac{\sin\theta}{\cos\theta} + \dfrac{\sin\frac{1}{2}\theta}{\cos\frac{1}{2}\theta}$

$$=\frac{\sin\theta\cos\frac{1}{3}\theta+\cos\theta\sin\frac{1}{3}\theta}{\cos\theta\cos\frac{1}{3}\theta}=\frac{\sin(\theta+\frac{1}{3}\theta)}{\cos\theta\cos\frac{1}{3}\theta}$$
$$=\frac{\sin\frac{4}{3}\theta}{\cos\theta\cos\frac{1}{3}\theta}=\frac{4\sin\frac{1}{3}\theta\cos\frac{1}{3}\theta\cos\frac{2}{3}\theta}{\cos\theta\cos\frac{1}{3}\theta}$$
$$=\frac{4\sin\frac{1}{3}\theta\cos\frac{2}{3}\theta}{\cos\theta},$$

and $\tan\theta-\tan\frac{1}{3}\theta=\dfrac{\sin\theta}{\cos\theta}-\dfrac{\sin\frac{1}{3}\theta}{\cos\frac{1}{3}\theta}=\dfrac{\sin\frac{2}{3}\theta}{\cos\theta\cos\frac{1}{3}\theta}=\dfrac{2\sin\frac{1}{3}\theta}{\cos\theta}$;

therefore $\tan^2\theta-\tan^2\frac{1}{3}=\dfrac{8\sin^2\frac{1}{3}\theta\cos\frac{2}{3}\theta}{\cos^2\theta}$.

Second Solution. *By Mr. R. S. Howland, St. Paul's College.*

$$\cos\tfrac{1}{3}\theta+\cos\theta=2\cos\tfrac{1}{2}(\theta+\tfrac{1}{3}\theta)\cos\tfrac{1}{2}(\theta-\tfrac{1}{3}\theta)$$
$$=2\cos\tfrac{2}{3}\theta\cos\tfrac{1}{3}\theta,$$

and $\cos\frac{1}{3}\theta-\cos\theta=2\sin\frac{2}{3}\theta\sin\frac{1}{3}\theta$
$$=4\sin^2\tfrac{1}{3}\theta\cos\tfrac{1}{3}\theta;$$

and multiplying these equations, member by member,
$$\cos^2\tfrac{1}{3}\theta-\cos^2\theta=8\sin^2\tfrac{1}{3}\theta\cos\tfrac{2}{3}\theta\cos^2\tfrac{1}{3}\theta;$$
dividing by $\cos^2\frac{1}{3}\theta\cos^2\theta$,
$$\sec^2\theta-\sec^2\tfrac{1}{3}\theta=\frac{8\sin^2\frac{1}{3}\theta\cos\frac{2}{3}\theta}{\cos^2\theta},$$
$$=\tan^2\theta-\tan^2\tfrac{1}{3}\theta.$$

Third Solution. *By Mr. J. Blickensderfer, jun., Canal Dover, Ohio.*

In the known formula
$$\tan^2 a-\tan^2 b=\frac{\sin(a+b)\sin(a-b)}{\cos^2 a\cos^2 b},$$
take $a=\theta$, $b=\frac{1}{3}\theta$; then we have
$$\tan^2\theta-\tan^2\tfrac{1}{3}\theta=\frac{\sin\frac{4}{3}\theta\sin\frac{2}{3}\theta}{\cos^2\theta\cos^2\frac{1}{3}\theta}$$
$$=\frac{4\sin\frac{1}{3}\theta\cos\frac{1}{3}\theta\cos\frac{2}{3}\theta\,.\,2\sin\frac{1}{3}\theta\cos\frac{1}{3}\theta}{\cos^2\theta\cos^2\frac{1}{3}\theta}$$
$$=\frac{8\sin^2\frac{1}{3}\theta\cos\frac{2}{3}\theta}{\cos^2\theta}.$$

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(38). Question VIII. *By the Editor.*

In a plane triangle, given that
$$b=a\sin C,\quad c=a\cos B.$$
Find its angles.

First Solution. *By Mr. Warren Colburn, St. Paul's College.*

By a known relation among the sides and angles of a plane triangle,
$$c=a\cos B+b\cos A;$$
but in the triangle in question,

$$c = a \cos B;$$

therefore, $0 = b \cos A$, or $\cos A = 0$, and $A = \frac{1}{2}\pi$.

Similarly, $b = a \cos C + c \cos A$
$= a \cos C$, since $A = \frac{1}{2}\pi$,
$= a \sin C$, by the question;

then $\sin C = \cos C$, and $C = \frac{1}{4}\pi$.

Finally $B = \pi - (A + C) = \pi - \frac{3}{4}\pi = \frac{1}{4}\pi$,

and the triangle is right-angled and isosceles.

SECOND SOLUTION. *By Mr. B. Birdsall.*

Multiply the second given equation by $2c$, then

$$2c^2 = 2ac \cos B,$$

but, in all triangles, $b^2 = a^2 + c^2 - 2ac \cos B$;

hence, by addition, $b^2 + c^2 = a^2$,

and therefore the triangle is right-angled, a being the hypothenuse; then $A = 90^\circ$, and $B + C = 90^\circ$; consequently

$$\sin C = \cos B,$$

or $$\frac{b}{a} = \frac{c}{a}, \text{ and } b = c;$$

or the triangle is isosceles, and $B = C = 45^\circ$.

THIRD SOLUTION. *By Omicron, jun., Chapel Hill, N. C.*

Here $1 : \sin C :: a : b :: \sin A : \sin B$,

or $\sin B = \sin A \sin C$ . . . . (1),

also, $1 : \cos B :: a : c :: \sin A : \sin C$,

or $\sin C = \sin A \cos B$ . . . . (2).

But $\sin C = \sin (A + B) = \sin A \cos B + \cos A \sin B$,

and $0 = \cos A \sin B$;

therefore, either $\sin B = 0$, and $B = 0$, or 180°, which cannot be;

or $\cos A = 0$, and $A = 90^\circ$.

Then (1) becomes $\sin B = \sin C$,

or $B = C = 45^\circ$.

—— Mr. Coaklay favored us with two solutions to this question, similar to the second and third of the above; the solution of L. was similar to the second, and that of Mr. Campbell to the third.

(39). QUESTION IX. *By* ——.

Let $a_1, a_2, a_3, \ldots\ldots$, be the sides of any plane polygon, and $\varphi_1, \varphi_2, \varphi_3, \ldots\ldots$, the angles they severally make with any straight line in the same plane, all counted in the same direction; prove that

$$a_1 \sin \varphi_1 + a_2 \sin \varphi_2 + a_3 \sin \varphi_3 + \ldots\ldots = 0,$$
$$a_1 \cos \varphi_1 + a_2 \cos \varphi_2 + a_3 \cos \varphi_3 + \ldots\ldots = 0.$$

FIRST SOLUTION. *By Mr. D. D. Hughes.*

It is evident that $a_1 \sin \varphi_1 + a_2 \sin \varphi_2 + \ldots$, is the sum of the projections of all the sides of the polygon on a given line;—it is also clear that the sum of the projections of the sides of one-half of the po-

lygon is equal to the sum of the projections of the sides of the other half, while they have contrary signs, and therefore their sum = 0, or

$$a_1 \sin\varphi_1 + a_2 \sin\varphi_2 + a_3 \sin\varphi_3 + \ldots\ldots = 0,$$

Similarly, $a_1 \cos\varphi_1 + a_2 \cos\varphi_2 + \ldots\ldots$ is the sum of the projections of all the sides on a line perpendicular to the first one, and therefore

$$a_1 \cos\varphi_1 + a_2 \cos\varphi_2 + a_3 \cos\varphi_3 + \ldots\ldots = 0.$$

SECOND SOLUTION. *By Mr. B. Birdsall.*

Let a parallel to the given line be drawn through the vertex from which a_1 is counted, and the sides will make the same angles with this line as with the given one; then

$a_1 \sin\varphi_1$, $a_1 \sin\varphi_1 + a_2 \sin\varphi_2$, $a_1 \sin\varphi_1 + a_2 \sin\varphi_2 + a_3 \sin\varphi_3$, &c. will represent the distances of the successive vertices of the polygon from this line; but when we have reckoned entirely round the polygon and arrived at the starting point, its distance from the line passing through it = 0, and therefore

$$a_1 \sin\varphi_1 + a_2 \sin\varphi_2 + a_3 \sin\varphi_3 + \ldots\ldots = 0.$$

In the same manner, $a_1 \cos\varphi_1 + a_2 \cos\varphi_2 + \ldots\ldots$, will express the distance of the same vertex to a line passing through it, perpendicular to the given line, and therefore it must = 0.

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(40). QUESTION X. *By* ——.

Having given the sum of the sides that include the right angle of a spherical triangle, and the difference of their opposite angles; to determine the sides and angles of the triangle.

FIRST SOLUTION. *By Mr. R. S. Howland.*

$$\cos(\mathrm{B}-\mathrm{A}) = \cos\mathrm{B}\cos\mathrm{A} + \sin\mathrm{B}\sin\mathrm{A} \quad \ldots\ldots \quad (1),$$

$$\cos(a+b) = \cos a\cos b - \sin a\sin b \quad \ldots\ldots \quad (2).$$

But $\cos a\cos b = \cos h$, h being the hypothenuse,

and $\sin a = \sin h\sin\mathrm{A}$, $\sin b = \sin h\sin\mathrm{B}$ ...... (3),

therefore $\sin a\sin b = \sin^2 h\sin\mathrm{A}\sin\mathrm{B}$, and (2) becomes

$$\cos(a+b) = \cos h - \sin^2 h\sin\mathrm{A}\sin\mathrm{B},$$

or $\cos h - \cos(a+b) = \sin^2 h\sin\mathrm{A}\sin\mathrm{B}$.

Divide (1) by this equation, recollecting that $\cot\mathrm{A}\cot\mathrm{B} = \cos h$, then

$$\frac{\cos(\mathrm{B}-\mathrm{A})}{\cos h - \cos(a+b)} = \frac{\cos h + 1}{\sin^2 h} = \frac{1}{1-\cos h};$$

whence $$1 - \cos h = \frac{1-\cos(a+b)}{1+\cos(\mathrm{B}-\mathrm{A})},$$

or $$2\sin^2\tfrac{1}{2}h = \frac{\sin^2\frac{1}{2}(a+b)}{\cos^2\frac{1}{2}(\mathrm{B}-\mathrm{A})},$$

and $$\sin\tfrac{1}{2}h = \frac{\sin\frac{1}{2}(a+b)}{\cos\frac{1}{2}(\mathrm{B}-\mathrm{A})}\cdot\sqrt{\tfrac{1}{2}},$$

whence h is found. Again

$$\cos(a+b) + \cos(b-a) = 2\cos a\cos b = 2\cos h,$$

therefore $\sin^2\frac{1}{2}(a+b) + \sin^2\frac{1}{2}(b-a) = 2\sin^2\frac{1}{2}h = \dfrac{\sin^2\frac{1}{2}(a+b)}{\cos^2\frac{1}{2}(\mathrm{B}-\mathrm{A})}$,

and $\sin^2\frac{1}{2}(b-a)=\dfrac{\sin^2\frac{1}{2}(a+b)}{\cos\frac{1}{2}(B-A)}-\sin^2\frac{1}{2}(a+b)=\sin^2\frac{1}{2}(a+b)\tan^2\frac{1}{2}(B-A)$

or $\quad \sin\frac{1}{2}(b-a)=\sin\frac{1}{2}(a+b)\tan\frac{1}{2}(B-A)$,

whence $b-a$ is found, and consequently a and b. Moreover

$$\cos h=\cot A\cot B=\frac{\cos A\cos B}{\sin A\sin B}=\frac{\cos(B-A)+\cos(B+A)}{\cos(B-A)-\cos(B+A)},$$

and $$1-\cos h=\frac{\sin^2\frac{1}{2}(a+b)}{\cos^2\frac{1}{2}(B-A)}=\frac{-2\cos(B+A)}{\cos(B-A)-\cos(B+A)};$$

whence $$\cos(A+B)=\frac{\cos(B-A)\sin^2\frac{1}{2}(a-b)}{\sin^2\frac{1}{2}(a+b)-2\cos^2\frac{1}{2}(B-A)},$$

from which $B+A$ is found, and thence A and B.

Second Solution. *By L., Murray Co., Geo.*

Let C be the right angle, then we shall have

$$\sin c\cos A=\cos a\sin b,\qquad \sin c\sin A=\sin a,$$
$$\sin c\cos B=\cos b\sin a,\qquad \sin c\sin B=\sin b,$$
$$\therefore \sin c(\cos A+\cos B)=\sin(a+b),\quad \therefore \sin c(\sin A-\sin B)=\sin a-\sin b;$$

by division, $$\frac{\sin A-\sin B}{\cos A+\cos B}=\frac{\sin a-\sin b}{\sin(a+b)};$$

But $$\frac{\sin A-\sin B}{\cos A+\cos B}=\tan\tfrac{1}{2}(A-B),$$

also, $$\sin a-\sin b=2\sin\tfrac{1}{2}(a-b)\cos\tfrac{1}{2}(a+b),$$

and $$\sin(a+b)=2\sin\tfrac{1}{2}(a+b)\cos\tfrac{1}{2}(a+b);$$

hence $$\tan\tfrac{1}{2}(A-B)=\frac{\sin\frac{1}{2}(a-b)}{\sin\frac{1}{2}(a+b)},$$

or $$\sin\tfrac{1}{2}(a-b)=\sin\tfrac{1}{2}(a+b)\tan\tfrac{1}{2}(A-B).$$

From this equation $a-b$ can be determined, and thence a and b; the side c and the angles A and B, are then found from the usual formulas.

—— Most of our correspondents solved this question by means of Napier's Analogies.

(41). Question XI. *By θ.*

Given the equation

$$y^2-yx+1=0,$$

to express y, by the method of Indeterminate Co-efficients, in a series of monomials arranged 1°. according to the ascending powers of x, 2°. according to the descending powers of x.

Solution. *By Δ.*

Make $y=Ax^a+Bx^b+Cx^c+$ &c.; then, by substitution,

$$\begin{aligned}0&=y^2-yx+1\\&=A^2x^{2a}+2ABx^{a+b}+B^2x^{2b}+2ACx^{a+c}+2BCx^{b+c}+C^2x^{2c}+\&c.\\&\quad -Ax^{a+1}-Bx^{b+1}-Cx^{c+1}-Dx^{d+1}-\&c.\\&\quad +x^0;\end{aligned}$$

and among these exponents, we can make

1°. $2a=0$, or $a=0$; $a+b=a+1$, or $b=1$, $a+c=b+1$, or $c=2$, &c.; hence

$$y=\mathrm{A}+\mathrm{B}x+\mathrm{C}x^2+\mathrm{D}x^3+\&c.$$

$$0=y^2-yx+1$$

$$\begin{array}{rl|l|l|l}
=\mathrm{A}^2 & +2\mathrm{AB} & x+\mathrm{B}^2 & x^2+2\mathrm{AD} & x^3+\mathrm{C}^2 \\
+1 & -\mathrm{A} & +2\mathrm{AC} & +2\mathrm{BC} & +2\mathrm{AE} \\
 & & -\mathrm{B} & -\mathrm{C} & +2\mathrm{BD} \\
 & & & & -\mathrm{D}
\end{array}\Bigg| x^4+\&c.$$

Then

$$\begin{array}{rll}
\mathrm{A}^2+1=0, & \text{or} & \mathrm{A}=\pm\sqrt{-1}, \\
2\mathrm{AB}-\mathrm{A}=0, & \text{or} & \mathrm{B}=\tfrac{1}{2}, \\
\mathrm{B}^2+2\mathrm{AC}-\mathrm{B}=0, & \text{or} & \mathrm{C}=\dfrac{1}{2.4\mathrm{A}}, \\
2\mathrm{AD}+2\mathrm{BC}-\mathrm{C}=0, & \text{or} & \mathrm{D}=0, \\
2\mathrm{AE}+2\mathrm{BD}-\mathrm{D}+\mathrm{C}^2=0, & \text{or} & \mathrm{E}=\dfrac{1}{2.4.4^2.\mathrm{A}}, \\
\&c. & & \&c.
\end{array}$$

and $y=\pm\sqrt{-1}+\frac{1}{2}x\mp\dfrac{\sqrt{-1}}{2.4}x^2\mp\dfrac{\sqrt{-1}}{2.4.4^2}x^4\mp\dfrac{1.3.\sqrt{-1}}{2.4.6.4^3}x^6\ \ldots\ldots$

2°. $2a=a+1$, or $a=1$, $a+b=b+1=0$, or $b=-1$, $c+1=2b$, $c=-3$, &c.;

$$y=\mathrm{A}x+\mathrm{B}x^{-1}+\mathrm{C}x^{-3}+\mathrm{D}x^{-5}+\&c.$$

$$0=y^2-yx+1$$

$$\begin{array}{r|l|l|l}
=\mathrm{A}^2 & x^2+2\mathrm{AB} & x^0+2\mathrm{AC} & x^{-2}+2\mathrm{AD} \\
-\mathrm{A} & -\mathrm{B} & +\mathrm{B}^2 & +2\mathrm{BC} \\
 & +1 & -\mathrm{C} & -\mathrm{D}
\end{array}\Bigg| x^{-4}+\&c.$$

Then

$$\begin{array}{rl}
\mathrm{A}^2-\mathrm{A}=0, & \text{and } 2\mathrm{A}-1=\pm 1,\ \mathrm{A}=\tfrac{1}{2}(1\pm 1), \\
2\mathrm{AB}-\mathrm{B}+1=0, & \mathrm{B}=\mp 1, \\
2\mathrm{AC}+\mathrm{B}^2-\mathrm{C}=0, & \mathrm{C}=\mp 1=\mp\dfrac{2}{2}, \\
2\mathrm{AD}+2\mathrm{BC}-\mathrm{D}=0, & \mathrm{D}=\mp 2=\mp\dfrac{2.6}{2.3}, \\
2\mathrm{AE}+2\mathrm{BD}+\mathrm{C}^2-\mathrm{E}=0, & \mathrm{E}=\mp 5=\mp\dfrac{2.6.10}{2.3.4}, \\
2\mathrm{AF}+2\mathrm{BE}+2\mathrm{CD}-\mathrm{F}=0, & \mathrm{F}=\mp 14=\mp\dfrac{2.6.10.14}{2.3.4.5}, \\
\&c. & \&c.
\end{array}$$

and $y=\frac{1}{2}(1\pm 1)x\mp x^{-1}\mp\dfrac{2}{2}x^{-3}\mp\dfrac{2.6}{2.3}x^{-5}\mp\dfrac{2.6.10}{2.3.4}x^{-7}\ \ldots\ldots$

Cor. $\dfrac{2.6.10\ldots.2(2n-3)}{1.2.3\ldots\ldots\ldots n}=\dfrac{\overline{n+1}.\overline{n+2}\ldots.\overline{2n-2}}{1.2.3\ldots\ldots.\overline{n-1}}=$ a whole number, for all values of n.

QUESTION XII. *By* ——.

The equation of a plane is

$$\mathrm{A}x+\mathrm{B}y+\mathrm{C}z+\mathrm{D}=0;$$

prove that the area of the triangle intercepted on the plane by the three rectangular co-ordinate planes, is

$$\frac{\mathrm{D}^2}{2\mathrm{ABC}}.\sqrt{\mathrm{A}^2+\mathrm{B}^2+\mathrm{C}^2},$$

FIRST SOLUTION. *By L., Murray Co., Geo.*

The distances from the origin to the intersections of the plane with the axes of x, of y, and of z, are

$$x' = -\frac{D}{A},\ y' = -\frac{D}{B},\ z' = -\frac{D}{C};$$

the trace (a) on the plane of xy is

$$a = \sqrt{x'^2 + y'^2} = \frac{D}{AB}\sqrt{A^2 + B^2};$$

the perpendicular (p) from the origin upon the trace (a), is

$$p = \frac{x'y'}{a} = \frac{D}{\sqrt{A^2 + B^2}};$$

and the perpendicular (P) from the intersection of the plane with the axis of z upon the trace (a), is

$$P = \sqrt{p^2 + z'^2} = \frac{D}{C}.\sqrt{\frac{A^2 + B^2 + C^2}{A^2 + B^2}};$$

hence the area of the triangle is

$$\tfrac{1}{2}aP = \frac{D^2}{2ABC}\sqrt{A^2 + B^2 + C^2}\ .$$

SECOND SOLUTION. *By Mr. Geo. W. Coaklay.*

From the given equation we have, for the intersections of the plane with the axes $x = -\frac{D}{A}$, $y = -\frac{D}{B}$, $z = -\frac{D}{C}$. Put a, b, c for the sides of the triangle, then, evidently

$$a^2 = x^2 + y^2 = \frac{D^2}{A^2} + \frac{D^2}{B^2},\ b^2 = y^2 + z^2 = \frac{D^2}{B^2} + \frac{D^2}{C^2},\ c^2 = x^2 + z^2 = \frac{D^2}{A^2} + \frac{D^2}{C^2}.$$

But, if s be the area of the triangle, it is easy to prove that

$$4s^2 = a^2b^2 - \tfrac{1}{4}(a^2 + b^2 - c^2)^2;$$

now $$a^2b^2 = \frac{D^4}{B^4} + \frac{D^4}{A^2B^2} + \frac{D^4}{A^2C^2} + \frac{D^4}{B^2C^2},$$

and $$a^2 + b^2 - c^2 = \frac{2D^2}{B^2};$$

hence, $$4s^2 = \frac{D^4}{A^2B^2} + \frac{D^4}{A^2C^2} + \frac{D^4}{B^2C^2}$$

$$= \frac{D^4}{A^2B^2C^2}(A^2 + B^2 + C^2),$$

and $$2s = \frac{D^2}{ABC}\sqrt{A^2 + B^2 + C^2}.$$

—— *Cor.* If α, β, γ, be the angles which a perpendicular (δ) from the origin upon the plane, makes with the three axes respectively, its equation will be

$$x\cos\alpha + y\cos\beta + z\cos\gamma = \delta,$$

the area becomes $\quad s = \tfrac{1}{2}\delta^2 \sec\alpha \sec\beta \sec\gamma,$

and the tretraedron whose surfaces are the given plane and the three co-oridinate planes has for its volume

$$\tfrac{1}{6}\delta^3 \sec\alpha \sec\beta \sec\gamma.$$

List of Contributors to the Junior Department, and of Questions answered by each. The figures refer to the number of the questions, as marked in Number VI., Article XIV., page 359.

W. B. Benedict, Upperville, Fanquier Co., Va., answered 2, 3, 4.
B. Birdsall, New-Hartford, N. Y., ans. all the questions.
J. Blickensderfer, jun., Canal Dover, Ohio, ans. 3, 4, 7, 8.
J. V. Campbell, Sophomore Class, St. Paul's College, ans. 1, 3, 8.
Geo. W. Coaklay, Peekskill Academy, N. Y., ans. all the questions.
Warren Colburn, Sophomore Class, St. Paul's College, ans. 3, 4, 8.
E. H. Delafield, Freshman Class, St. Paul's College, ans. 1, 3, 4, 5, 8.
Delta, ans. 11.
R. S. Howland, Junior Class, St. Paul's College, ans. 7, 8, 10, 12.
D. D. Hughes, Syracuse Academy, N. Y., 3, 5, 6, 7, 9, 12.
L., Murray County, Geo., ans. all the questions.
Omicron, jun., Chapel Hill, N. C., ans. all but 11.

ARTICLE III.

QUESTIONS TO BE ANSWERED IN NUMBER VIII.

Their solutions must arrive before August 1st, 1839.

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(43). QUESTION I. *By* ——.

Transform the numbers 25 and 389 into a system of notation whose base, or scale of relation is 3; multiply them in that state, exhibiting the process, and transform the result to the decimal scale.

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(44). QUESTION II. *By* ——.

Prove that

$$(n+1)(n+2)\;\ldots\;(n+n)=2^n\times 1\,.\,3\,.\,5\;\ldots\;(2n-1).$$

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(45). QUESTION. III. *By L., Murray Co., Geo.*

To find x, y, z, there are given the three equations

$$(1),\quad xz=y^2,$$
$$(2),\quad 2xz-x-y-z=1,$$
$$(3),\quad x^2+y^2+z^2=3x+3y+3z.$$

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(46). QUESTION IV. *By Mr. E. H. Delafield.*

Find what relation must have place among the co-efficients of the equation

$$x^4+\text{A}x^3+\text{B}x^2+\text{C}x+\text{D}=0,$$

so that the process for taking away its second term, may also take away its fourth term at the same time.

(47). QUESTION V. *By* —.

Let $\mathrm{x}_0 = 1, 2\mathrm{x}_1 = x + \frac{1}{x}, 2\mathrm{x}_2 = x^2 + \frac{1}{x^2}, 2\mathrm{x}_3 = x_3 + \frac{1}{x^3}$, &c.; prove that

$$\mathrm{x}_{n+1} + \mathrm{x}_{n-1} = 2\mathrm{x}_n\mathrm{x}_1.$$

(48). QUESTION VI. *By* —.

Adapt the relations of the sides and angles of a plane triangle to the case where the sides are in arithmetical progression, and find the area of the triangle.

(49.) QUESTION VII. *By* β.

Prove that, θ being any angle,

$$\operatorname{cosec} \theta - 2 \operatorname{cosec} 3\theta = \cot 3\theta \sec \theta.$$

(50). QUESTION VIII. *By* —.

In Navigation, find the bearing and distance from a given place on the earth's surface to another one, differing from the former 10° in latitude and 10° in longitude.

(51). QUESTION IX. *By* —.

The earth being supposed a perfect sphere, draw a great circle arc between any two points on the surface which differ from each other 10° in latitude and 10° in longitude; find its length, and the angle it makes with the meridian of either place.

(52). QUESTION X. *By* —.

Find the points of intersection of the two ellipses

$$7y^2 + 4x^2 = 28,$$
$$6y^2 + 5x^2 = 30,$$

related to the same axes of co-ordinates, and determine the angles they make with each other at these points.

(53). QUESTION XI. *By Mr. H. Clay, Eng.*

Find, when $\varphi = 0$, the value of the expression

$$\frac{1}{\varphi^2} - \frac{1}{\tan^2 \varphi}.$$

(54). QUESTION XII. *By* —.

AB is the diameter of a given circle, and C any point in the circumference; from C let fall CD perpendicular to AB, and upon it take CP = AD; find the curve in which the point P is always found.

SENIOR DEPARTMENT.

ARTICLE I.

SOLUTIONS TO THE QUESTIONS PROPOSED IN NUMBER V.

(82). QUESTION I. *By an Engineer.*

The following is an extract from my Note-Book:

No.	Bearing.	Distance.	Elevation.
1	N. 10° 15′ E.	27,64 ch.	+ 17° 54′
2	N. 28° 40′ W.	100,00	+ 20° 19′
3	N. 20° 00′ W.	15,00	+ 7° 43′
4	N. 20° 00′ W.	37,26	— 5° 26′
5	N. 30° 17′ E.	68,75	— 11° 13′

It is required to find the Bearing, Distance, and Elevation of a line drawn from the beginning of the first to the end of the fifth line, by a method applicable to all practical cases of the kind. The Bearing is considered as the inclination of a vertical plane through the two places with the plane of the meridian.

FIRST SOLUTION. *By Mr. R. S. Howland.*

We may suppose these to be six points in space, referred to three rectangular axes of x, y, z; the plane of xz being the plane of the meridian, that of yz the vertical east and west plane, and that of xy the horizontal plane through the first point. The formulas for calculation will then be the common ones for transformation from polar to rectangular co-ordinates: thus

Altitude = Distance × sin. Elevation,

Diff. lat. = Distance × cos. Elevation × cos. Bearing,

Departure = Distance × cos. Elevation × sin. Bearing.

Then the altitude of the last point above the first is the sum of the several altitudes taken with their proper signs; the difference of latitude and departure, are the sums of the difference of latitudes and departures of the several distances; and their bearing, elevation, and distance are calculated from the converse formulas

$$\text{tan Bearing} = \frac{\text{Departure}}{\text{Diff. Lat.}},$$

$$\text{tan. Elevation} = \frac{\text{Altitude} \times \text{cos. Bearing}}{\text{Diff. Lat.}},$$

$$\text{Distance} = \frac{\text{Altitude}}{\text{sin.Elevation}}.$$

No.	Bearing.	Distance.	Elevation.	Altitude.	Diff. lat. N.	Departure W.
1	N. 10° 15′ E.	27,54	17° 54′	8,46	25,79	— 4,66
2	N. 28° 40′ W.	100,00	20° 19′	34,72	82,28	44,99
3	N. 20° 00′ W.	15,00	7° 43′	2,01	13,97	5,08
4	N. 20° 00′ W.	37,26	—5° 26′	— 3,53	34,70	12,63
5	N. 36° 17′ E.	68,75	—11° 13′	—13,37	54,36	—39,91
Total	N. 4° 54½′ W.	213,74	7° 36⅓′	28,29	211,10	18,13

(83). Question II. *By Mr. J. F. Macully, Esq., N. Y.*

It is required to draw a chord through the focus of a given ellipse, which shall divide the area in a given ratio.

First Solution. *By the Proposer.*

The equation of the ellipse, the focus being the pole, is

$$r = \frac{\mathrm{B}^2}{\mathrm{A} - c\cos\varphi}:$$

and since the area, cut off by a focal chord making the angle θ with the angular axis, is to be in a given ratio with the whole area,

$$n\mathrm{AB}\pi = \tfrac{1}{2}\int_\theta^{\pi+\theta} r^2 d\varphi = \tfrac{1}{2}\mathrm{B}^4\int_\theta^{\pi+\theta}\frac{d\varphi}{(\mathrm{A} - c\cos\varphi)^2}$$

$$= \mathrm{AB}\cot^{-1}\left(\frac{\mathrm{C}}{\mathrm{B}}\sin\theta\right) - \frac{\mathrm{AB}^2 c\sin\theta}{\mathrm{A}^2 - c^2\cos^2\theta}.$$

Let, then, ψ be an angle such that $\cot\frac{1}{2}\psi = \frac{\mathrm{C}}{\mathrm{B}}\sin\theta$, then

$$\psi - \sin\psi = 2n\pi.$$

If ψ_1 be an approximate root of this equation, the correction to be applied in order to obtain a nearer value, will be

$$\delta\psi = \frac{2n\pi - (\psi_1 - \sin\psi_1}{2\sin^2\frac{1}{2}\psi_1)}.$$

Having found ψ from this equation, the angle θ is determined by

$$\sin\theta = \frac{\mathrm{B}}{\mathrm{C}}\cot\tfrac{1}{2}\psi.$$

Second Solution. *By Mr. B. Birdsall.*

The equation of the ellipse, referred to the transverse axis and a diameter making the angle a with it as axes of co-ordinates, is

$$(\mathrm{A}^2\sin^2 a + \mathrm{B}^2\cos^2 a)y^2 + 2\mathrm{B}^2\cos a\,xy + \mathrm{B}^2x^2 = \mathrm{A}^2\mathrm{B}^2,$$

and if we put $\mathrm{A}^2\sin^2 a + \mathrm{B}^2\cos^2 a = k^2\sin^2 a$, we shall have

$$y\sin a = -\frac{\mathrm{B}^2}{k^2}\cot a x \pm \frac{\mathrm{AB}}{k^2}\sqrt{k^2 - x_2};$$

that is, if y' and y'' are the segments into which a chord parallel to the axis of y is divided by the axis of x,

$$y' \sin a = -\frac{B^2}{k^2} \cot a \,.\, x + \frac{AB}{k^2}\sqrt{k^2 - x^2},$$

$$y'' \sin a = +\frac{B^2}{k^2} \cot a \,.\, x + \frac{AB}{k^2}\sqrt{k^2 - x^2};$$

and if m represent the given ratio, the area cut off by a chord through the focus, parallel to the axis of y, is

$$m\text{AB}\pi = \int_{Ae}^{A} (y'dx \sin a + y''dx \sin a)$$

$$= \int_{Ae}^{A} \frac{2\text{AB}}{k^2} \,.\, dx\sqrt{k^2 - x^2},$$

$$= \text{AB} \tan^{-1} \frac{A\sqrt{k^2 - A^2e^2} - Ae\sqrt{k^2 - A^2}}{A^2e + \sqrt{(k^2 - A^2)(k^2 - A^2e^2)}} + \frac{A^2B}{k^2}\sqrt{k^2 - A^2}$$

$$+ \frac{A^2B}{k^2}\sqrt{k^2 - A^2} - \frac{A^2Be}{k^2}\sqrt{k^2 - A^2e^2},$$

and restoring the value of $k^2 = A^2 + B^2 \cot^2 a$,

$$m\pi = \tan^{-1} \frac{\text{AB}(1 - e \cos a)}{A^2 e \sin a + B^2 \cot a} + \frac{\text{AB}(\cot a - e \operatorname{cosec} a)}{A^2 + B^2 \cot^2 a},$$

from which the angle a may be found.

THIRD SOLUTION. *By Mr. O. Root, Syracuse Academy.*

Let a, b be the semiaxes of the ellipse, c the distance from the centre to the focus. If the circle, radius a, whose projection is the ellipse, be divided by a chord into two parts having the given ratio, the projected areas will have the same ratio; this will be the case when

$$\psi - \tfrac{1}{2} \sin 2\psi : \pi - \psi + \tfrac{1}{2} \sin 2\psi = m : n$$

or $$2\psi - \sin 2\psi = \frac{2m\pi}{m+n},$$

ψ being the angle included by the greater segment, and $m : n$ the given ratio. Having found ψ by this equation, let θ' and β be the angles the dividing line and its projection make with the axis, we have

$$c \sin \theta' = a \cos \psi, \text{ or } \sin \theta' = \frac{a}{c} \cos \psi,$$

and $$\tan \beta = \frac{b}{a} \tan \theta'.$$

The question is impossible if $\cos \psi > \dfrac{c}{a}$, or $\sin \psi < \dfrac{b}{a}$,

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(84). QUESTION III. *By Investigator.*

Find the *polar* equation of a straight line on a plane; and bring it to the form best adapted to general use. Apply it to finding the equation of a tangent to the ellipse at any point, the pole being at the focus and the angular axis the line of the foci.

SOLUTION. *By Prof. C. Avery, Hamilton College.*

Let p be the perpendicular from the origin on the line, α the angle made by p with the angular axis, r the radius vector, ω the angle r makes with the axis; then

$$r\cos(\omega-\alpha)=p, \quad \text{or } r=p\sec(\omega-\alpha) \quad . \quad . \quad . \quad . \quad (1).$$

is the polar equation of the straight line. If b be the intercept of the angular axis, between the origin and the line, and β the angle the line makes with the axis, we shall have

$$\alpha+\beta=\tfrac{1}{2}\pi, \quad p=b\sin\beta=b\cos\alpha \quad . \quad . \quad . \quad . \quad . \quad (2),$$

which may be used instead of p and α, if more covenient.

If $r'\omega'$, $r''\omega''$ be two points in any curve, we shall have for the secant passing through them

$$r'=p\sec(\omega'-\alpha), \quad r''=p\sec(\omega''-\alpha)$$

and
$$\frac{r'-r''}{r'}=\frac{\sec(\omega'-\alpha)-\sec(\omega''-\alpha)}{\sec(\omega'-\alpha)}=\frac{\cos(\omega''-\alpha)-\cos(\omega'-\alpha)}{\cos(\omega''-\alpha)}$$
$$=\frac{2\sin\frac{1}{2}(\omega'-\omega'')\sin\frac{1}{2}(\omega'+\omega''-2\alpha)}{\cos(\omega''-\alpha)},$$

and, ultimately, when this secant becomes a tangent,

$$\frac{dr'}{r'}=d\omega'\tan(\omega'-\alpha),$$

$$\text{or,} \quad \alpha=\omega'-\tan^{-1}\frac{dr'}{r'd\omega'}, \text{ and } p=r'\cos(\omega'-\alpha) \quad . \quad . \quad . \quad (3);$$

the equation of the tangent is then

$$r\cos(\omega-\alpha)=r'\cos(\omega'-\alpha).$$

But
$$\omega-\alpha=(\omega-\omega')+(\omega'-\alpha),$$
and
$$\cos(\omega-\alpha)=\cos(\omega-\omega')\cos(\omega'-\alpha)-\sin(\omega-\omega')\sin(\omega'-\alpha),$$
$$\therefore \quad r\{\cos(\omega-\omega')-\sin(\omega-\omega')\tan(\omega'-\alpha)\}=r',$$
$$\text{or} \quad r\left\{\cos(\omega-\omega')-\sin(\omega-\omega')\frac{dr'}{r'd\omega'}\right\}=r' \quad . \quad . \quad . \quad . \quad . \quad (4)$$

is the equation of the tangent to a given curve at the point $r'\omega'$.

For the ellipse,
$$\frac{A(1-e^2)}{r'}=1+e\cos\omega',$$
and
$$\frac{dr'}{r'd\omega'}=\frac{e\sin\omega'}{1+e\cos\omega}.$$

Hence the equation of a tangent to the ellipse is

$$r\{\cos(\omega-\omega')+e\cos\omega\}=A(1-e^2) \quad . \quad . \quad . \quad . \quad (5).$$

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(85). QUESTION IV. *By Mr. P. Barton, jun.*

The co-ordinates of the vertex of a cone of revolution are

$$x=-4, \quad y=3, \quad z=-2;$$

the equations of its axis are

$$x=\tfrac{1}{2}z-3, \quad y=-\tfrac{1}{3}z+2\tfrac{1}{3};$$

and its vertical angle is 90°. It is required to find where its surface is intersected by the line whose equations are

$$x=z+6, \quad y=-z-5.$$

First Solution. *By Prof. M. Catlin, Hamilton College.*

Let the vertex of the cone be xyz, a point in the given line at the distance D from the vertex, be $x''y''z''$, and a point in the axis at the distance D$'$ from the first, be $x'y'z'$; then, from the equations of the line,

$$\begin{aligned} D^2 &= (x''-x)^2+(y''-y)^2+(z''-z)^2 \\ &= (z''+10)^2+(z''+8)^2+(z''+8)^2 \\ &= 3z''^2+40z''+162 \quad \ldots\ldots\ldots \quad (1). \end{aligned}$$

Also,

$$\begin{aligned} D'^2 &= (x'-x'')^2+(y'-y'')^2+(z'-z'')^2 \\ &= (z''-\tfrac{1}{2}z'+9)^2+(z''-\tfrac{1}{3}z'+7\tfrac{1}{3})^2+(z'-z'')^2 \\ &= \tfrac{49}{36}z'^2-\tfrac{11}{3}z'z''+3z''^2-\tfrac{125}{9}z'+\tfrac{28}{3}z''+134\tfrac{7}{9} \quad (2). \end{aligned}$$

If D$'$ be perpendicular to the axis, we shall have

$$\frac{dD'}{dz'}=\tfrac{49}{18}z'-\tfrac{11}{3}z''-\tfrac{125}{9}=0, \text{ or } z'=\frac{66z''+250}{49} \quad (3),$$

and

$$D'^2=\frac{26z''^2+684z''+4868}{49} \quad \ldots\ldots\ldots \quad (4).$$

Now in order that the point $(x''y''z'')$ may be in the surface of the cone whose vertical angle is 90°, we must have $D^2=2D'^2$, or

$$49(3z''^2+40z''+168)=2(26z''^2+684z''+4868),$$

$$95z''^2+592z''=1504;$$

therefore

$$\begin{aligned} z''&=-8{,}16947, &\text{or } z''&=1{,}93789, \\ x''&=-2{,}16947, &\text{or } x''&=7{,}93789, \\ y''&=3{,}16947, &\text{or } y''&=-6{,}93789, \end{aligned}$$

are the co-ordinates of the two points of intersection.

Second Solution. *By Mr. Geo. R. Perkins, Clinton Liberal Institute.*

The equation of any line passing through the vertex of the cone is

$$x+4=a(z+2),\quad y-3=b(z+2);$$

and if this be a linear element of the cone whose vertical angle $=90°$,

$$\cos 45° = \sqrt{\tfrac{1}{2}} = \frac{1+aa'+bb'}{\sqrt{1+a^2+b^2}.\sqrt{1+a'^2+b'^2}},$$

or

$$(1+a^2+b^2)(1+a'^2+b'^2)=2(1+aa'+bb')^2;$$

and substituting the values

$$a=\frac{x+4}{z+2},\quad b=\frac{y-3}{z+2},\quad a'=\tfrac{1}{2},\ b'=-\tfrac{1}{3},$$

we get the equation of the surface of the cone

$$23z^2-41y^2-31x^2-48yz+72xz-24xy+524z+54y-32x+379=0.$$

Combining this with the equations of the given straight line, we have, for the points of intersection,

$$\begin{aligned} x&=7{,}9378, &\text{or } x&=-2{,}1694, \\ y&=-6{,}9378, &\text{or } y&=3{,}1694, \\ z&=1{,}9378, &\text{or } z&=8{,}1694. \end{aligned}$$

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(86). Question X. *By* ψ.

The circumference of a circle is divided into n equal parts, and from the points of division perpendiculars are drawn upon a given diameter of the circle. If lines be drawn from any given point in the plane of the circle to the points where these perpendiculars intersect the diameter, it is required to find the sum of the squares of these lines.

SOLUTION. *By Mr. O. Root.*

Let $r =$ radius of the circle, $a =$ distance of the given point from the centre, $\varphi =$ angle between a and the fixed diameter, $r\theta$ the arc between the first point and the extremity of the fixed diameter, and $r\theta + r \cdot \frac{2m\pi}{n}$ will be the arc between any other one and that extremity; then a general expression for the square of any of the required lines is

$$a^2 - 2ar\cos\varphi\cos\left(\theta + \frac{2m\pi}{n}\right) + r^2\cos^2\left(\theta + \frac{2m\pi}{n}\right)$$

$$= a^2 + \tfrac{1}{2}r^2 - 2ar\cos\varphi\cos\left(\theta + \frac{2m\pi}{n}\right) + \tfrac{1}{2}r^2\cos\left(2\theta + \frac{4m\pi}{n}\right),$$

m having all the integral values from 0 to $n-1$, or from 1 to n inclusive; hence the sum required is

$$n(a + \tfrac{1}{2}r^2) - 2ar\cos\varphi\left\{\cos\theta + \cos\left(\theta + \frac{2\pi}{n}\right) + \ldots \cos\left(\theta + \frac{2(n-1)\pi}{n}\right)\right\}$$

$$+ \tfrac{1}{2}r^2\left\{\cos 2\theta + \cos\left(2\theta + \frac{4\pi}{n}\right) + \ldots \cos\left(2\theta + \frac{4(n-1)\pi}{n}\right)\right\}.$$

But $\cos\theta + \cos\left(\theta + \frac{2\pi}{n}\right) + \cos\left(\theta + \frac{4\pi}{n}\right) + \ldots \cos\left(\theta + \frac{2(n-1)\pi}{n}\right) = 0,$

and $\cos 2\theta + \cos\left(2\theta + \frac{4\pi}{n}\right) + \cos\left(2\theta + \frac{8\pi}{n}\right) + \ldots \cos\left(2\theta + \frac{4(n-1)\pi}{n}\right) = 0$:

hence the sum is $\quad n(a^2 + \tfrac{1}{2}r^2)$;

except when $n=2$, and then,

$$\cos 2\theta + \cos\left(2\theta + \frac{4\pi}{n}\right) + \ldots \cos\left(2\theta + \frac{4(n-1)\pi}{n}\right) = 2\cos 2\theta,$$

and the sum is $\quad 2(a^2 + r^2\cos^2\theta)$.

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(87). QUESTION VI. *By* ——

Def. In the ellipse or hyperbola, the parameter of any diameter is that chord of the system it bisects which is a third proportional to that diameter and its conjugate.

It is required

1°. To find what diameters may properly be said to have parameters.

2°. To find the locus of the middle points of all the parameters of the same curve.

3°. Having given a parameter, to find if possible, another one perpendicular to it.

SOLUTION. *By Mr. Geo. R. Perkins.*

Let a and b be the semi-axes of the ellipse or hyperbola, a' and b' any two semi-conjugate diameters, making the angles w and w' with a, r the the distance of the parameter from the centre, measured on a' and $2p$ the parameter. Then

1°. $\quad a'^2p^2 \pm b'^2r^2 = \pm a'^2b'^2,$

or, since $\quad a'^2p^2 = b'^4,\ r^2 = a'^2 \mp b'^2 \quad . \quad . \quad . \quad . \quad (1),$

so that, in the ellipse, when the upper signs are used, the diameter has

a parameter, only when $a'>b'$, or when it is included between the equal conjugates. In the hyperbola it is limited by the circumstance that p is only real when a' is real, and therefore those diameters only, included between the assympotes have parameters.

2°. $$a'^2 = \frac{\pm a^2 b^4}{a^2 \sin^2 w \pm b^2 \cos^2 w},\quad b'^2 = \frac{a^2 b^2}{a^2 \sin^2 w' \pm b^2 \cos^2 w'};$$

but $$\tan w \tan w' = \mp \frac{b^2}{a^2};$$

therefore $$\sin^2 w' = \frac{b^4 \cos^2 w}{a^4 \sin^2 w + b^4 \cos^2 w},\quad \cos^2 w' = \frac{a^4 \sin^2 w}{a^4 \sin^2 w + b^4 \cos^2 w},$$

and $$b'^2 = \pm \frac{a^4 \sin^2 w + b^4 \cos^2 w}{a^2 \sin^2 w \pm b^2 \cos^2 w};$$

$$\therefore \qquad r^2 = (a^2 \mp b^2) . \frac{-a^2 \sin^2 w \pm b^2 \cos^2 w}{a^2 \sin^2 w \pm b^2 \cos^2 w} \quad \ldots\ldots \quad (2),$$

which is the polar equation of the locus. Its rectangular equation is

$$(x^2 + y^2)(a^2 y^2 \pm b^2 x^2) = (a^2 \mp b^2)(-a^2 y^2 \pm b^2 x^2) \ldots\ldots (3).$$

The curve passes through the foci; for the ellipse it is of the form of the *Lemniscata*, the centre being the multiple point; for the hyperbola, the assymptotes are also those of the curve.

3°. If there are two parameters at right angles to each other, the diameters to which they are parallel are so. Now if β be the angle the equal conjugates of the ellipse, or the assymptotes of the hyperbola make with the transverse axis, so that $\tan \beta = \frac{b}{a}$, we have seen that the limits of w are $-\beta$ and β, and the limits of w' are β and $\pi - \beta$, then, if there are parameters perpendicular to each other, they must be parallel to diameters within the limits $w' = \beta$ and $w' = \frac{1}{2}\pi - \beta$, or $w' = \frac{1}{2}\pi + \beta$ and $w' = \pi - \beta$; and if the angle w' of the given parameter be within these limits, we shall have the angle of its perpendicular $= \frac{1}{2}\pi + w'$, and for corresponding diameter

$$\tan w = \mp \frac{b^2}{a^2} \cot(\tfrac{1}{2}\pi + w') = \pm \frac{b^2}{a^2} \tan w'$$

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(88). Question VII. By ——.

The theorem of M. Sturm, published in the "Memoirs présèntes par des Savans Estrangers," for 1835, may be stated thus:

Let $\mathrm{x} = 0$, be any algebraical equation, whose co-efficients are real, and whose roots are unequal, and let $\mathrm{x}_1 = \frac{d\mathrm{x}}{dx}$. Apply to the two polynomials x, x_1 the process for finding their greatest common measure, the several remainders having all their signs changed from $+$ to $-$, and from $-$ to $+$, before they are used as new divisors, and in that state let them be represented by $\mathrm{x}_2, \mathrm{x}_3, \mathrm{x}_4 \ldots\ldots\ldots \mathrm{x}_m$. In the series of polynomials

$$\mathrm{x}, \mathrm{x}_1, \mathrm{x}_2, \mathrm{x}_3, \ldots\ldots \mathrm{x}_m,$$

which are of continually decreasing dimensions in x, x_m being independent of x, let any two numbers p and q be successively substituted for x, noting the signs of the two series of results. *Then the difference between*

the number of variations of the first series of signs, and that of the second, expresses exactly the number of real roots of the given equation, which are comprised between the two numbers p and q.

It required to apply this theorem to the general equation

$$x^4 + ax^2 + bx + c = 0,$$

in order to determine the number and nature of its real roots.

SOLUTION. *By Prof. Peirce, Cambridge University.*

We have

$$\begin{aligned} \mathrm{x} &= x^4 + ax^2 + bx + c, \\ \mathrm{x}_1 &= 4x^3 + 2ax + b, \\ \mathrm{x}_2 &= -2ax^2 - 3bx - 4c, \\ \mathrm{x}_3 &= -(2a^3 + 9b^2 - 8ac)x - (a^2b + 12bc) = -a'x - b', \\ \mathrm{x}_4 &= 2ab'^2 - 3a'bb' + 4a'^2c = a''. \end{aligned}$$

By substituting for x $\quad x = +\infty$,
the values become, when none are deficient,

$$\mathrm{x} = +\infty, \mathrm{x}_1 = +\infty. \mathrm{x}_2 = -a\infty, \mathrm{x}_3 = -a'\infty, \mathrm{x}_4 = a'';$$

substituting for x $\quad x = -\infty$,
they become $\mathrm{x} = \infty$, $\mathrm{x}_1 = -\infty$, $\mathrm{x}_2 = -a\infty$, $\mathrm{x}_3 = a'\infty$, $\mathrm{x}_4 = a''$;
substituting for $\quad x = 0$,
they become $\mathrm{x}=c$, $\mathrm{x}_1=b$, $\mathrm{x}_2 = -4c$, $\mathrm{x}_3 = -b'$, $\mathrm{x}_4 = a''$.

In order that all the roots may be real, we must then have

$$a < 0, \quad a' < 0, \quad a'' > 0;$$

and if moreover, we have $\quad c > 0$,
two of the roots are positive and two are negative.

But if $c < 0$ we must obviously have $bb' > 0$ in order that $a'' > 0$, and therefore, $\quad (a^2 + 12c)b^2 > 0$, or $c > -\frac{1}{12}a^2$.
If, then, $b > 0$, one of the roots is positive and three are negative; and if $b < 0$, three roots are positive and one is negative.

In all other cases but the preceding, the equation has two real roots and no more when a'' is negative, and none when a'' is positive.

If c is also negative, one of the roots is positive and the other negative.

If c is positive the two real roots must both have the opposite sign to b.

Particular Cases.

1. When $b = 0$ the roots are all real, when $c > 0$ and $< \frac{1}{4}a^2$, two being positive and two negative; but if $c > \frac{1}{4}a^2$ they are all imaginary. If $c < 0$, two of the roots are real, the one being positive, the other negative.
2. When $c = -\frac{1}{12}a^2$, there are two real roots, one of which is positive, the other negative.
3. When $c = 0$, it is reduced to a cubic equation.
4. When $a'' = 0$, the equation has a pair of equal roots each of which is
$$x = -\frac{b'}{a'},$$
and the other roots are easily found.
5. When $a' = 0$, and a positive, the equation has no real roots, but if a is negative it has two real roots, which, when c is negative are, the

one positive, the other negative; but when c is positive, they both have the same sign as $-b$. If a is zero, the equation is reduced to $x^4+c=0$,

6. When $a=0$, there are no real roots if b and $256c^3-27b^4$ are both positive or both negative. But if b is positive and $256c^3-27b^4$ negative, one of the two real roots is positive and the other negative; and if b is negative while $256c^3-27b^4$ is positive, both the real roots are positive. If $256c^3-27b^4=0$, the equation has two real roots each equal to $-\frac{4c}{3b}$.

(89). QUESTION VIII. *By Prof. B. Peirce, Harvard University.*

Prove that if all the roots of the equation

$$x^n-\text{A}x^{n-2}+\text{B}x^{n-3}-\&c.=0,$$

are real that we shall have

$$n(n-1)(3\text{B})^2<(n-2)^2(2\text{A})^3.$$

FIRST SOLUTION. *By the Proposer.*

Put
$$x=x'+a$$
in the given equation, and it becomes

$$x'^n+nax'^{n-1}+\left[\frac{n(n-1)}{2}a^2-\text{A}\right]x'^{n-2}+\left[\frac{n(n-1)(n-2)}{2.3}a^3-(n-2)\text{A}a+\text{B}\right]x'^{n-3}+\&c.=0;$$

and if we take for a such a value as to satisfy the equation

$$\frac{n(n-1)}{2}a^2-\text{A}=0,$$

we reduce it to

$$x'^n+nax'^{n-1}+[\text{B}-\tfrac{2}{3}.(n-2)\text{A}a]x'^{n-3}+\&c.=0,$$

in which, since a term is wanting, the co-efficients preceding and following it must not have the same sign, or all the roots would not be real as they are in the given equation, and as they must be in this equation also, since A must be positive, and therefore a is real. The quotient of these co-efficients must then be negative, or

$$\frac{\text{B}}{a}<\tfrac{2}{3}(n-2)\text{A},$$

or $\quad 3\text{B}<2(n-2)\text{A}a.$

Squaring $\quad (3\text{B})^2<(n-2)^2a^2(2\text{A})^2,$

and by substitution $\quad (3\text{B})^2<\frac{(n-2)^2(2\text{A})^3}{n(n-1)},$

or $\quad n(n-1)(3\text{B})^2<(n-2)^2(2\text{A})^3;$

which, for the cubic equation, is

$$3.2\,(3\text{B})^2<(2\text{A})^3, \text{ or } (\tfrac{1}{2}\text{B})^2<(\tfrac{1}{3}\text{A})^3;$$

for the biquadratic,

$$4.3\,(3\text{B})^2<4(2\text{A})^3, \text{ or } \text{B}^2<(\tfrac{2}{3}\text{A})^3;$$

for the 5th degree,

$$5.4\,(3\text{B})^2<3^2(2\text{A})^3, \text{ or } 5\text{B}^2<2\text{A}^3,$$

and for n very large $\quad (3\text{B})^2<(2\text{A})^3.$

SECOND SOLUTION. *By Prof. M. Catlin, Hamilton College.*

Let a, b, c, e, &c., be the roots of the given equation; then, since the second term is wanting, we shall have the following relations:

$$a+b+c+e+\&c.=0 \quad \ldots \ldots (1),$$
$$a^2+b^2+c^2+e^2+\&c.=2\text{A} \quad \ldots \ldots (2),$$
$$a^3+b^3+c^3+e^3+\&c.=3\text{B} \quad \ldots \ldots (3).$$

(See Hutton's Math. Vol. 2. p. 262, Cor. 5.) Therefore the given inequality becomes

$$(n-2)^2(a^2+b^2+c^2+e^2+\&c.)^3>n(n-1)(a^3+b^3+c^3+e^3+\&c.)^2 \quad (4),$$

or

$$(n-2)^2(a^2+b^2+c^2+e^2+\&c.)^3-n(n-1)(a^3+b^3+c^3+e^3+\&c.)^2=+\text{P} \quad (5)$$

P being essentially positive. To prove (5) we will find the minimum value of P. Differentiate (5), eliminate da by means of (1), and equate the co-efficients of db, dc, &c., separately with zero, and we easily get

$$\left.\begin{array}{l}(n-2)^2(2\text{A})^2(a-b)-n(n-1)(3\text{B})(a^2-b^2)=0\\(n-2)^2(2\text{A})^2(a-c)-n(n-1)(3\text{B})(a^2-c^2)=0\\(n-2)^2(2\text{A})^2(a-e)-n(n-1)(3\text{B})(a^2-e^2)=0\\\quad\&c. \qquad\qquad\qquad\qquad \&c.\end{array}\right\} \quad \ldots (6).$$

Hence, b, c, e, &c., are obviously equal to each other, and therefore any one of them must by (1)

$$=-\frac{a}{n-1} \quad \ldots \ldots \ldots (7).$$

By virtue of (7) equation (5) becomes

$$\text{P}=(n-2)^2\left(a^2+\frac{a^2}{n-1}\right)^3-n(n-1)\left(a^3-\frac{a^3}{(n-1)^2}\right)^2=0 \ . \ (8).$$

Therefore (5) becomes

$$(n-2)^2(2\text{A})^3=n(n-1)(3\text{B})^2 \quad \ldots \ (9),$$

when $n-1$ of the roots of the given equation are equal to each other. In all other cases we shall have $\text{P}>0$, and consequently

$$(n-2)^2(2\text{A})^3>n(n-1)(3\text{B})^2 \ . \ . \ . \ . \ (10).$$

Cor. Whenever we have the equation $(n-2)^2(2\text{A})^3=n(n-1)(3\text{B})^2$, we may infer that $n-1$ of the roots are equal to each other.

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(90). QUESTION IX. *By Prof. F. N. Benedict, University of Vt.*

To determine the locus of the intersection of two tangents or normals to the common parabola which include an angle whose tangent varies as a given function of the co-ordinates of the point of intersection.

SOLUTION. *By Prof. Avery.*

I. Let the equation of the parabola be $y^2=2mx$; the equations of the tangents at the points $y'x'$, $y''x''$, are

$$yy'=m(x+x')=mx+\tfrac{1}{2}y'^2,\ yy''=m(x+x'')=mx+\tfrac{1}{2}y''^2 \quad . \ (1),$$

hence, at their point of intersection, xy,

$$y(y'-y'')=\tfrac{1}{2}(y'^2-y''^2),\ \text{or}\ y'+y''=2y \quad . \ . \ (2);$$

and $$y(y'+y'')=2mx+\tfrac{1}{2}(y'+y'')^2-y'y'',\ \text{or}\ y'y''=2mx \ (3).$$

Since $\frac{m}{y'}$ and $\frac{m}{y''}$ are the tangents of their inclinations with the axis of x, the tangent of their mutual inclination is

$$\frac{\frac{m}{y''}-\frac{m}{y'}}{1+\frac{m^2}{y'y''}}=\frac{m(y'-y'')}{y'y''+m^2}=\frac{2\sqrt{y^2-2mx}}{2x+m}=f(x,y),$$

and $$2\sqrt{y^2-2mx}=(2x+m)f(x,y) \quad . \quad . \quad . \quad . \quad (4),$$

is the equation of the locus required.

II. The equations of the normals are

$$y-y'=-\frac{y'}{m}(x-x') \text{ or } m^2y=m(m-x)y'+\tfrac{1}{2}y'^3 \quad . \quad . \quad (5),$$

and $$y-y''=-\frac{y''}{m}(x-x''), \text{ or } m^2y=m(m-x)y''+\tfrac{1}{2}y''^3 \quad . \quad . \quad (6);$$

and the tangent of their angle of intersection is

$$\frac{m(y''-y')}{y'y''+m^2}=f'(x,y) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7).$$

And if y', y'' be eliminated among equations (5), (6), (7), the result will be the equation of the curve.

Equations (5) and (6) show that, when

$$27y^2<2m(m-x)^3,$$

there are three real values of y', and three of y'', or that from any point within the evolute of the parabola, three normals can be drawn to the parabola.

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(91). QUESTION X. *By Wm. Lenhart, Esq., York. Penn.*

Having given a series of whole numbers whose third order of differences are constant, and of which a given term is divisible by a given prime number m; it is required to find that term in the series which is divisible by m^n, n being a given whole number.

FIRST SOLUTION. *By the Proposer.*

Let $a_0, a_1, a_2 \ldots a_{n'}$ represent the given series; $A+Bn'+Cn'^2+Dn'^3$, a general expression for any term of the series, A, B, C and D being constants to be determined from the first four terms of the series, and $a_{n'}$ a term divisible by m. Then

$$a_{n'}=A+Bn'+Cn'^2+Dn'^3=A_1m \quad . \quad . \quad . \quad . \quad . \quad (1).$$

In (1) write $n'+n''m$ for n', and

$$a_{n'+n''m}=A+(n'+n''m)B+(n'+n''m)^2C+(n'+n''m)^3D;$$

from which

$$a_{n'+n''m}=\left\{\begin{array}{l} A+n'B+n''mB \\ \quad n'^2C+2n'n''mC+n''^2m^2C \\ \quad n'^3D+3n'^2n''mD+3n'n''^2m^2D+''^3m^3D \end{array}\right\},$$

$$=\begin{cases} A_1 m + n''m(B + n'(2C + 3n'D)) \\ \quad + n''^2 m^2 (C + 3n'D) \\ \quad + n''^3 m^3 D\,; \end{cases}$$

Or putting $B + n'(2C + 3n'D) = s$, and $C + 3n'D = s'$, and dividing by m

$$\frac{1}{m}\cdot a_{n'+n''m} = A_1 + n''s + n''^2 ms' + n''^3 m^2 D = A_2 m \quad . \quad . \quad (2).$$

Now find a value of n'' as directed at the end of the solution, that (2) may divide by m, and make the division; then

$$\frac{1}{m^2}\cdot a_{n'+n''m} = A_2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3).$$

We may here remark that since A_1, in (2), is prime to m, s must be so also; for otherwise, (2) could not be made divisible by m, and in that case the problem would be impossible.

Again, for n'', in (2), write $n'' + n'''m$, and it will become

$$\frac{1}{m}\cdot a_{n'+n''m+n'''m^2} = A_1 + (n''+n'''m)s + m(n''+n'''m)^2 s' + m^2(n''+n'''m)^3 D\,;$$

from which, by a development as above, we shall have

$$\frac{1}{m}\cdot a_{n'+n''m+n'''m^2} = \begin{cases} A_1 + n''s + n'''ms \\ \quad n''^2 ms' + 2n''n'''m^2 s' + n'''^2 m^2 s' \\ \quad n''^3 m^2 D + 3n''^2 n'''m^3 D + 3n''n'''^2 m^4 D + n'''^3 m^5 D \end{cases}$$

$$=\begin{cases} A_2 m + n'''m(s + n''m(2s' + 3n''mD)) \\ \quad + n'''^2 m^3 (s' + 3n''mD) \\ \quad + n'''^3 m^5 D\,; \end{cases}$$

Or, restoring the values of s and s', and dividing by m,

$$\frac{1}{m^2}\cdot a_{n'+n''m+n'''m^2} = \begin{Bmatrix} A_2 + n'''[B + n'(2C + 3n'D) + n''m(2C + 6n'D + 3n''mD)] \\ + m^2 n'''^2 (C + 3n'D + 3n''mD) \\ + n'''^3 m^4 D \end{Bmatrix}$$

$$= A_3 m \quad . \quad . \quad . \quad (4).$$

Or, finding n''', as directed, and dividing by m,

$$\frac{1}{m^3}\cdot a_{n'+n''m+n'''m^2} = A_3 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5).$$

Again, by writing $n''' + n''''m$ for n''', in (4), and developing as before, we shall find

$$\frac{1}{m^3}\cdot a_{n'+n''m+n'''m^2+n''''m^3} = \begin{Bmatrix} A_3 + n''''[B + n'(2C + 3n'D) + n''m(2C + 6n'D + 3n''mD) \\ + n'''m^2(2C + 6n'D + 6n''mD + 3n'''m^2D)] \\ + n''''^2 m^3 (C + 3n'D + 3n''mD + 3n'''m^2D) + n''''^3 m^6 D \end{Bmatrix}$$

$$= A_4 m \quad . \quad . \quad (6).$$

Or, finding n'''', and dividing by m,

$$\frac{1}{m^4}\cdot a_{n'+n''m+n'''m^2+n''''m^3} = A_4 \quad . \quad . \quad . \quad . \quad . \quad . \quad (7).$$

We shall now have generally,

$$\frac{1}{m^{n-1}}\cdot a_{n'+n''m\ldots n^{(n)}m^{n-1}} = A_{n-1} + n^{(n)}(B + 2Cp + 3Dp^2)$$
$$+ n^{(n)2} m^{n-1}(C + 3Dp) + n^{(n)3} m^{2n-2} D = A_n m\,. \qquad (8);$$

wherein $$p = (n' + n''m \ldots n^{(n-1)}m^{n-2}).$$
And to find a value of $n^{(n)}$ that shall render (8) divisible by m, we prefer, among several, the following simple method, namely:—Let the remainder of $A_{n-1} \div m$ be denoted by R, and the remainder of s or $(B + n'(2C + 3n'D)) \div m$ by R$'$ which is constant, as the nature of the process or inspection alone plainly indicates. Then from

$$R + n^{(n)}R' = mT \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9),$$

we have $n^{(n)}$, and thence dividing (8) by m, we obtain finally

$$\frac{1}{m^n} \cdot a_{n'+n''m \ldots n^{(n)}m^{n-1}} = A_n,$$

$$\text{or,} \qquad a_{n'+n''m \ldots n^{(n)}m^{n-1}} = A_n m^n = \text{term required} \quad . \quad . \quad (10).$$

We may now readily perceive the general law or system which pervades the whole subject, and consequently be enabled to resolve, not only each particular case of each example, but also the general problem itself, namely, when the n^{th} differences are constant.

Example. Given 13, 14, 17, 23, &c.

Appl. Let $a_0 = A = 13$; $a_1 = A + B + C + D = 14$; $a_2 = A + 2B + 4C + 8D = 17$; $a_3 = A + 3B + 9C + 27D = 23$; then $A = 13$, $B = \frac{1}{3}$, $C = \frac{1}{2}$, and $D = \frac{1}{6}$. Assume $m = 7$, then $n' = 1$, $A_1 = 2$, and $s = 1\frac{5}{6}$.

Remainder of $A_1 \div m = R = 2$, and remainder of $s \div m = R' = \frac{5}{6}$ constant. Then from

(9), $n'' = 4$, and (2) becomes 644: therefore $A_2 = 92$. Rem. $A_2 \div m = R = 1$,
(9), $n''' = 2$, " (4) " 7133: " $A_3 = 1019$. " $A_3 \div m = R = 4$,
(9), $n'''' = 1$, " (6) " 50771: " $A_4 = 7253$. " $A_4 \div m = R = 1$,
(9), $n''''' = 2$, " (8) &c., &c.

Now, if we assume $n = 3$, we shall have from (10)

$$a_{127} = a_3 m^3 = 1019 \times 343 = 349517 = \text{term required.}$$

SECOND SOLUTION. *By the same gentleman.*

Let the given series be denoted by $a_0, a_1, a_2 \ldots . a_n$; a general expression for any term in the series by $A + Bn' + Cn'^2 + Dn'^3$; in which A, B, C, and D are constants to be determined by the first four terms of the series, and let $a_{n'}$ be a term divisible by m. Then

$$a_{n'} = A + n'(B + n'(C + n'D)) = A_1 m \quad . \quad . \quad . \quad . \quad (1).$$

In (1), write $n' + n''m$ for n', and

$$\begin{aligned} a_{n'+n''m} &= A + (n'+n''m)(B + (n'+n''m)(C + n'D + n''mD)) \\ &= A + (n'+n''m)(B + n'(C+n'D) + n''m(C + 2n'D + n''mD)) \\ &= A_1 m + n''m\left(\frac{A_1 m - A}{n'} + (n'+n''m)(C + 2n'D + n''mD)\right), \end{aligned}$$

by comparing terms with their values deduced from (1). Or, reducing further, putting $\dfrac{A_1 m - A}{n'} + n'(C + 2n'D) = s$, and dividing by m,

$$\frac{1}{m} \cdot a_{n'+n''m} = A_1 + n'(s + n''m(C + 3n'D + n''mD)) = A_2 m \quad . \quad . \quad (2).$$

Find n'', as directed at the end of the solution, to make (2) divide by m, and effect the division: then

$$\frac{1}{m^2} \cdot a_{n'+n''m} = \text{A}_2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3).$$

We may here remark that since A_1, in (2), is prime to m, s must be so too, else (2) could not be made to divide by m, and in that case the problem would be impossible.

For n'' write $n'' + n'''m$, in (2), and it will become

$$\frac{1}{m} \cdot a_{n'+n''m+n'''m^2} = \text{A}_1 + (n'' + n'''m)(s + m(n'' + n'''m)(\text{C} + 3n'\text{D} + n''m\text{D} + n'''m^2\text{D})).$$

From which, reducing as above, and restoring the values of s and s', we shall have

$$\frac{1}{m^2} \cdot a_{n'+n''m+n'''m^2} = \text{A}_2 + n'''\left(\frac{\text{A}_2 m - \text{A}_1}{n''} + m(n''+n'''m)(\text{C}+3n'\text{D}+2n''m\text{D} + n'''m^2\text{D})\right) = \text{A}_3 m \quad . \quad . \quad (4).$$

Or, finding n''', as directed, and dividing by m,

$$\frac{1}{m^3} \cdot a_{n'+n''m+n'''m^2} = \text{A}_3 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5).$$

In the same way, writing $n''' + n''''m$ for n''', in (4), and reducing, &c., we shall find

$$\frac{1}{m^3} \cdot a_{n'+n''m+n'''m^2+n''''m^3} = \text{A}_3 + n''''\left(\frac{\text{A}_3 m - \text{A}_2}{n'''} + m^2(n'''+n''''m)(\text{C} + 3n'\text{D}+3n''m\text{D}+2n'''m^2\text{D}+n''''m^3\text{D})\right) = \text{A}_4 m \,.\, (6).$$

$$\frac{1}{m_4} \cdot a_{n'+n''m+n'''m^2+n''''m^3} = \text{A}_4 \quad . \quad . \quad . \quad . \quad (7).$$

And thence generally

$$\frac{1}{m^{n-1}} \cdot a_{n'+''nm\ldots n^{(n)}m^{n-1}} = \text{A}_{n-1} + n^{(n)}\left(\frac{\text{A}_{n-1} - \text{A}_{n-2}}{n^{(n-1)}} + m^{n-2}(n^{(n-1)} + n^{(n)}m)(\text{C}+3\text{D}p+m^{n-2}(2n^{(n-1} + n^{(n)}m)\text{D})\right) = \text{A}_n m \quad . \quad (8);$$

wherein $\quad p = (n' + n''m \; . \; . \; . \; . \; n^{(n-2)}m^{n-3}$.

To find $n^{(n)}$ that (8) may divide by m, we shall use this simple method, namely:—Let the remainder of $\text{A}_{n-1} \div m$ be denoted by R, and the remainder of $s \div m$ by R′ which is constant, as is plainly indicated by the process. Then from

$$\text{R} + n^{(n)}\text{R}' = m\text{T} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

we have $n^{(n)}$, and thence, dividing (8) by m, obtain finally

$$\frac{1}{m^n} \cdot a_{n'+n''m\ldots n^{(n)}m^{n-1}} = \text{A}_n;$$

or, $\quad a_{n'+n''m\ldots n^{(n)}m^{n-1}} = \text{A}_n m^n = \text{term required (10).}$

Note. A solution after the manner of that which we have given in Speculation No. 2, p. 331, Vol. I. Miscel., will be found to be exceedingly curious and interesting. If the series be represented by

$$\begin{array}{cccccccc} a'_0 & & a'_1 & & a'_2 & & a'_{n'} & \\ & b'_0 & & b'_1 & & b'_2 & & b'_{n'} \\ & & a & a+b & a+2b \ldots & a+n'b & & \\ & & & b & b & b & b & \end{array}$$

the general formulas, by the process of that solution, will be found to be

$a_0^{(n)} =$ first term of the n^{th} series.

$$b_0^{(n)} = b^{(n-1)}_{n^{(n-1)}} + \tfrac{1}{6} m^{n-2}(m-1)(3p + m^{n-2}(m+1)b). \quad (\text{A}),$$

$$m^{n-1}(p + m^{n-1}b) \quad \ldots\ldots\ldots \quad (\text{B}),$$

and $$m^{2n-2}b \quad \ldots\ldots\ldots \quad (\text{C}).$$

Also $a_n^{(n)}{}_{(n)} = a_0^{(n)} + n^{(n)}(b_0^{(n)} + \tfrac{1}{6}(n^{(n)}-1)m^{n-1}(3p + (n^{(n)}+1)m^{n-1}b))$ (D), which will divide by m.

$$b_n^{(n)}{}_{(n)} = b_0^{(n)} + \tfrac{1}{2} n^{(n)} m^{n-1}(2p + (n^{(n)}+1)m^{n-1}b) \quad \ldots \quad (\text{E}),$$

$$m^{n-1}(p + (n^{(n)}+1)m^{n-1}b) \quad \ldots\ldots \quad (\text{F}),$$

and $$m^{2n-2}b \quad \ldots\ldots\ldots\ldots \quad (\text{G}).$$

Wherein $p = (a - b \mp b(n' + n''m \ldots . n^{(n-1)}m^{n-2}))$; the remainder of $b_0^{(n)} \div m$ constant, and $n^{(n)}$ found from (9).

A, B, and C, are respectively the terms of the first, second and third order of differences corresponding to $a_0^{(n)}$, and

E, F, and G, those corresponding to $a_{n\,(n)}^{(n)}$.

THIRD SOLUTION. *By Dr. T. Strong, New-Brunswick, N. J.*

Let $\text{A} = m\text{A}'$ denote the term which is divisible by m, and let ΔA, $\Delta^2\text{A}$, $\Delta^3\text{A}$ be the first terms of the successive orders of differences, and mx the distance of another term from A, then the term is expressed by

$$m\text{A}' + mx\Delta\text{A} + \frac{mx(mx-1)}{2}\Delta^2\text{A} + \frac{mx(mx-1)(mx-2)}{2\,.\,3}.\Delta^3\text{A} = m^n p,$$

if this term be divisible by m^n, p being an integer, or putting

$$6\text{A}' = a,\ 6\Delta\text{A} - 3\Delta^2\text{A} + 2\Delta^3\text{A} = b,\ 3(\Delta^2\text{A} - \Delta^3\text{A}) = c,\ \Delta^3\text{A} = d,$$

then $$a + bx + mcx^2 + m^2dx^3 = 6pm^{n-} \quad \ldots \quad (1),$$

Put $$a + bx = my \quad \ldots\ldots\ldots \quad (2).$$

and by solving this equation as an indeterminate one, if v be the least value of x which satisfies the equation, $x = v + mx'$ will also satisfy it, and by substituting this value, and dividing by m, we get an equation of the form

$$a' + b'x' + mc'x'^2 + m^2x'^3 = qm^{n-2} \quad \ldots\ldots \quad (3),$$

q being an integer, and the exponent of m one less than in (1); putting then $a' + b'x' = my'$, and proceeding as for (2), we can successively lessen the exponent of m, until that exponent becomes zero.

This method is the same as Legendre's, who solves such equations in their most general form, in his *Theory of Numbers.*

—— The solutions of Messrs. Avery, Catlin, and Perkins were very complete, and we regret our inability to insert them. The theorem they deduce is comprised in the general expression in the third solution.

(92). QUESTION XI. *By J. F. Macully, Esq.*

Required the value of n terms of the continued product
$$(1+2\cos\theta)(1+2\cos 3\theta)(1+2\cos 9\theta)\ldots.$$

FIRST SOLUTION. *By Dr. Strong.*

Let $P=(1+2\cos\theta)(1+2\cos 3\theta)(1+2\cos 9\theta)\ldots.(1+2\cos 3^{x-1}\theta)$,
then, $$P+\Delta P = P(1+2\cos 3^x\theta,$$
and, $$\frac{\Delta P}{P} = 2\cos 3^x\theta = 3-4\sin^2 3^x.\tfrac{1}{2}\theta - 1$$
$$=\frac{\sin 3^{x+1}.\tfrac{1}{2}\theta}{\sin 3^x.\tfrac{1}{2}\theta}-1=\frac{\Delta\sin 3^x.\tfrac{1}{2}\theta}{\sin 3^x.\tfrac{1}{2}\theta},$$
which is satisfied by putting
$$P=\frac{\sin 3^x.\tfrac{1}{2}\theta}{\sin\tfrac{1}{2}\theta},$$
the product to x terms.

SECOND SOLUTION. *By Mr. B. Birdsall.*

$$\sin\tfrac{3}{2}\theta-\sin\tfrac{1}{2}\theta=2\sin\tfrac{1}{2}(\tfrac{3}{2}\theta-\tfrac{1}{2}\theta)\cos\tfrac{1}{2}(\tfrac{3}{2}\theta+\tfrac{1}{2}\theta)=2\sin\tfrac{1}{2}\theta\cos\theta;$$
therefore $$1+2\cos\theta=\frac{\sin\tfrac{3}{2}\theta}{\sin\tfrac{1}{2}\theta},$$
similarly $$1+2\cos 3\theta=\frac{\sin\tfrac{9}{2}\theta}{\sin\tfrac{3}{2}\theta},$$
&c.
$$\therefore (1+2\cos\theta)(1+2\cos 3\theta)\ldots.(1+2\cos 3^{n-1}\theta)$$
$$=\frac{\sin\tfrac{3}{2}\theta}{\sin\tfrac{1}{2}\theta}\times\frac{\sin\tfrac{9}{2}\theta}{\sin\tfrac{3}{2}\theta}\times\ldots.\times\frac{\sin 3^n.\tfrac{1}{2}\theta}{\sin 3^{n-1}.\tfrac{1}{2}\theta}$$
$$=\frac{\sin 3^n.\tfrac{1}{2}\theta}{\sin\tfrac{1}{2}\theta}.$$

THIRD SOLUTION. *By Mr. Geo. R. Perkins.*

If e be the base of the Naperian logarithms, and $x=\theta\sqrt{-1}$, we have
$$1+2\cos\theta=1+e^x+e^{-x},$$
$$1+2\cos 3\theta=1+e^{3x}+e^{-3x},\ \&c.$$
By actually multiplying two or three of the first factors, we shall perceive that the product is
$$1+e^x+e^{2x}+e^{3x}+\ldots.+e^{\frac{1}{2}(3^n-1)x}$$
$$+e^{-x}+e^{-2x}+e^{-3x}+\ldots.+e^{-\frac{1}{2}(3^n-1)x}$$
$$=1+\frac{e^{\frac{1}{2}(3^n+1)x}-e^x}{e^x-1}+\frac{e^{-\frac{1}{2}(3^n+1)x}+e^{-x}}{e^{-x}-1}$$
$$=\frac{e^{\frac{1}{2}(3^n-1)x}+e^{-\frac{1}{2}(3^n-1)x}-e^{\frac{1}{2}(3^n+1)x}-e^{-\frac{1}{2}(3^n+1)x}}{2-e^x-e^{-x}}$$
$$=\frac{\cos\tfrac{1}{2}(3^n-1)\theta-\cos\tfrac{1}{2}(3^n+1)\theta}{1-\cos\theta}=\frac{\sin\tfrac{1}{2}.3^n\theta}{\sin\tfrac{1}{2}\theta}.$$

—— Prof. Peirce, after paying a just compliment to the inventor of this beautiful question, proceeds thus:

"In general, $$\text{P}_n = \frac{\sin \frac{1}{2} \cdot 3^n\theta}{\sin \frac{1}{2}\theta}.$$

Case I. When $\theta = 4m\pi$, m being an integer, $\text{P}_n = 3^n$.

II. When $\theta = (2m+1) \cdot 2\pi$, $\text{P}_n = (-1)^n$.

III. When $\theta = (2m+1)\pi$, $\text{P}_n = 1$.

IV. When $3^n\theta = 2m'\pi$, but θ does not $= 2m\pi$, $\text{P}_n = 0$.

V. When $\theta = \dfrac{(4m+1)\pi}{3^n-1}$, $\text{P}_n = \cot \frac{1}{2}\theta$.

VI. When $\theta = \dfrac{4m\pi}{3^n-1}$, $\text{P}_n = 1$.

VII. When $\theta = \dfrac{(4m+2)\pi}{3^n-1}$, $\text{P}_n = -1$.

VIII. When $\theta = \dfrac{(4m+3)\pi}{3^n-1}$, $\text{P}_n = -\cot \frac{1}{2}\theta$.

IX. When $\theta = \dfrac{4m\pi}{3^n-2}$, $\text{P}_n = 2\cos \frac{1}{2}\theta$.

X. When $\theta = \dfrac{(4m+2)\pi}{3^n-2}$, $\text{P}_n = -2\cos \frac{1}{2}\theta$."

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(93). Question XII. *By Prof. B. Peirce.*

To find a curve whose radius of curvature is a given function of its arc.

First Solution. *By the Proposer.*

Let $s =$ the arc, $\varrho =$ the radius of curvature, $\varphi =$ the angle which ϱ makes with the axis of x, and let the given equation be

$$\varrho = \text{f}(s).$$

We have then $ds = \varrho d\varphi = \text{f}(s)d\varphi$, and $\varphi = \int \dfrac{ds}{\text{f}(s)}$.

Then $dx = ds \cdot \sin \varphi$, $\quad x = \int ds \cdot \sin \int \dfrac{ds}{\text{f}(s)}$.

$dy = ds \cdot \cos \varphi$, $\quad y = \int ds \cdot \cos \int \dfrac{ds}{\text{f}(s)}$.

Example I. Let $\text{f}(s) =$ constant $= \text{R}$; then $\int \dfrac{ds}{\text{f}(s)} = \dfrac{s}{\text{R}}$,

$$x = \int ds \sin \frac{s}{\text{R}} = -\text{R} \cos \frac{s}{\text{R}}, \quad y = \int ds \cos \frac{s}{\text{R}} = \text{R} \sin \frac{s}{\text{R}},$$

$$x^2 + y^2 = \text{R}^2,$$

which is the circle.

Example II. Let $\text{f}(s) = \text{A}s + \text{B}$, and we have $\int \dfrac{ds}{\text{f}(s)} = \dfrac{1}{\text{A}} . \log . (\text{A}s + \text{B})$.

If we put $\log(\text{A}s + \text{B}) = \text{A}s'$, $\text{A} = \tan \beta$;

$$x=\int ds \sin s'=\int ds' e^{As'} \sin s'=\cos \beta e^{As'} \sin (s'-\beta),$$

$$y=\int ds \cos s'=\int ds' e^{As'} \cos s'=\cos \beta e^{As'} \cos (s'-\beta).$$

Using the polar co-ordinates, r and θ, counting the angle θ from an axis inclined to that of x by angle β, we have

$$\tan(\theta-\beta)=\tan(s'-\beta), \quad \theta=s';$$

$$r=\cos\beta \,.\, e^{As'}=\cos\beta \,.\, e^{A\theta};$$

which is the equation of the logarithmic spiral.

Example III. Let $f(s)=\sqrt{1-s^2}$, then $\varphi=\int \frac{ds}{\sqrt{1-s^2}}=\text{arc sin.}\, s, s=\sin\varphi$;

$$x=\int ds \sin\varphi=\int d\varphi \cos\varphi \sin\varphi=\tfrac{1}{4}\cos 2\varphi,$$

$$y=\int ds \cos\varphi=\int d\varphi \quad \cos^2 \quad \varphi=\tfrac{1}{4}(2\varphi-\sin 2\varphi);$$

which are the equations of the cycloid.

SECOND SOLUTION. *By Prof. M. Catlin, Hamilton College.*

Let x and y be the co-ordinates of the required curve, s its length, r the radius of curvature, and z the angle formed by dx and ds.

Then $$r=\frac{dsdx}{d^2y} \quad \ldots\ldots\ldots \quad (1);$$

Also, $$dx=ds\cos z, \quad \text{and } dy=ds\sin z \quad \ldots\ldots \quad (2).$$

Hence, if ds is constant, $$d^2y=dsdz\cos z \quad \ldots\ldots\ldots \quad (3),$$

(1) becomes $$r=\frac{ds}{dz}, \text{ or } ds=rdz \quad \ldots\ldots\ldots \quad (4),$$

and (2) becomes $$dx=r\cos zdz, \; dy=r\sin zdz \quad \ldots\ldots \quad (5).$$

When the form of the function r is known, the required curve will be determined by (4) and (5).

For instance; Let $r=\sqrt{1-s^2}$; then, by (4), $s=\sin z$, and $r=\cos z$.

$$\therefore dx=\cos^2 zdz, \; dy=\sin z\cos zdz \quad \ldots\ldots \quad (6),$$

$$x=\tfrac{1}{4}\sin 2z+\tfrac{1}{2}z+c, \quad y=\tfrac{1}{2}\sin^2 z+c' \quad \ldots\ldots \quad (7),$$

which give the equation required, by the elimination of z.

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(94). QUESTION XIII. *By Prof. Peirce, Cambridge University.*

Find a curve which is its own involute.

SOLUTION. *By the Proposer.*

Let $s=$ the arc of the given curve,
$\varrho=$ its radius of curvature,
$\varphi=$ the angle which ϱ makes with a fixed axis;
and let s', ϱ', φ' be the corresponding quantities for the evolute whose fixed axis makes an angle $=a-90°$; with that of its involute.

We have, then,
$$d\varphi' = d\varphi, \quad \varphi' = \varphi + a,$$
$$ds' = \varrho' d\varphi' = \varrho' d\varphi = d\varrho,$$
$$\varrho' = \frac{d\varrho}{d\varphi};$$
so that if the curve is determined by the equation
$$\varrho = f(\varphi),$$
we have, for the evolute,
$$\varrho' = f(\varphi') = f(\varphi + a) = \frac{df(\varphi)}{d\varphi}.$$

If, now, we suppose
$$f(\varphi) = \text{A}e^{m\phi} + \text{A}'e^{m'\phi} + \&c.$$
we have A, A′, &c. arbitrary, and m, m', &c. the different roots of the equation,
$$e^{ma} = m, \text{ or } e^{ma} - m = 0.$$

The first member of this equation increases with m, when its differential co-efficient $ae^{ma} - 1$, is positive; and this differential co-efficient constantly increasing from $m = -\infty$ to $m = \infty$, is equal to zero, when
$$m = -\frac{\text{hyp. log. } a}{a}.$$

Case I. When $a > 0$, this value of m is real and corresponds to a minimum of the given first member, which then assumes the value
$$\frac{1 + \text{hyp. log. A}}{a}.$$
The given equation has, therefore,

no real root when hyp. log. $a > -1$, that is, when $a > \frac{1}{e}$;

it has two real roots when hyp. log. $a < -1$, that is, when $a < \frac{1}{e}$;

it has one real root when hyp. log. $a = -1$, that is, when $a = \frac{1}{e}$;

which is
$$m = e.$$

Case II. When $a < 0$, hyp. log. a is imaginary, and the differential co-efficient is always negative, so that the given equation has but one real root which is always positive.

Case III. When $a = 0$, the only possible root is $m = 1$.

The equation
$$e^{ma} - m = 0,$$
has, however, ***an infinite number of imaginary roots*** when a differs from zero, as we will now proceed to demonstrate. If we represent a pair of these roots by
$$g \pm h\sqrt{-1},$$
the corresponding terms, in the value of ϱ, may be reduced to the one
$$\text{B}e^{g\phi} \sin (h\varphi + \varepsilon),$$
in which B and ε are arbitrary constant quantities. And g and h are determined by the equation
$$g = h \cot ah,$$
$$\frac{h}{\sin ah} - e^{ah \cot ah} = 0.$$

Now if, in this last equation, $(n+\frac{1}{2})\pi$ be substituted for ah, its first member becomes

$$\frac{(n+\frac{1}{2})\pi}{a\cos n\pi}-1,$$

which, provided $\qquad (n+\frac{1}{2})\pi > a,$

has opposite signs for even and odd values of n, and that whether n or a is positive or negative; there is then, in general, a root of this equation between every two values

$$\frac{(n-\frac{1}{2})\pi}{a} \text{ and } \frac{(n+\frac{1}{2})\pi}{a},$$

n being any integer positive or negative; and thence a corresponding value of g from the first of these equations.

We may here observe that if $a=(2n+\frac{1}{2})\pi$, n being any integer positive or negative, we may take $h=1$, in which case we have $g=0$, and the term of ρ becomes

$$\text{B} \sin(\varphi+\varepsilon).$$

Having thus determined the equation

$$\rho=f(\varphi),$$

which contains an infinite number of arbitrary constant quantities, we have $\qquad x=-\int\rho d\varphi \,.\, \sin\varphi, \quad y=\int\rho d\varphi \,.\, \cos\varphi;$

and, on account of the infinite number of arbitrary constants, a curve which is its own involute may be found so as to pass through any points whatever.

Example 1. Suppose all the arbitrary constants, but one, to be zero, which one we will suppose to correspond to a real root of m. The curve is, in this case, the logarithmic spiral.

Example 2. Suppose $\quad a=(2n+\frac{1}{2})\pi,\ h=1,$

and suppose all the arbitrary constant quantities to be zero, but those which correspond to this value of h, we have

$$\rho=\text{B}\sin(\varphi+\varepsilon),$$

and by changing the fixed axis by an angle $=\varepsilon+90^\circ$, we have

$$\rho=\text{B}\sin(\varphi+90^\circ)=\text{B}\cos\varphi,$$

which corresponds to the cycloid.

Example 3. Suppose all but two of the constants to be zero, and these two to correspond to the real roots m and m'. We have then

$$\rho=\text{A}e^{m\phi}+\text{A}'e^{m'\phi}.$$

Let $\qquad m=\tan\beta,\ m'=\tan\beta',$

$$x=\text{A}\cos\beta e^{m\phi}\cos(\varphi+\beta)+\text{A}'\cos\beta' e^{m'\phi}\cos(\varphi+\beta'),$$

$$y=\text{A}\cos\beta e^{m\phi}\sin(\varphi+\beta)+\text{A}'\cos\beta' e^{m'\phi}\sin(\varphi+\beta');$$

and if we put

$$x_1=\text{A}\cos\beta e^{m\phi}\cos(\varphi+\beta),\ x_2=\text{A}'\cos\beta' e^{m'\phi}\cos(\varphi+\beta'),$$

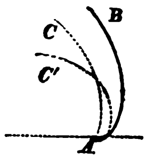
$$y_1=\text{A}\cos\beta e^{m\phi}\sin(\varphi+\beta),\ y_2=\text{A}'\cos\beta' e^{m'\phi}\sin(\varphi+\beta'),$$

we have $\qquad x=x_1+x_2,\quad y=y_1+y_2,$

in which x_1, y_1 are the co-ordinates of one logarithmic spiral, and x_2, y_2 those of another, so that if upon the two radius vectors of these spirals which correspond to the same value of φ, a parallelogram is formed, the diagonal drawn from the origin is the corresponding radius vector of the required curve. It appears, therefore, that when φ is very large, the

form of the curve is almost exactly like that of the spiral which corresponds to the larger value of m, and when φ is a very large negative quantity, the curve is almost identical with the other spiral. If A and A′ have opposite signs, the curve has a cusp of the first species corresponding to the value of φ

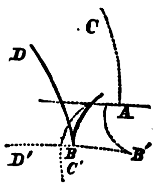
$$\varphi = \frac{\log. \text{A}' - \log. \text{A}}{(m - m') \log. e}.$$



Thus if $a = 18^\circ$, we have

$$m = 5.3212,$$
$$m' = 1,7127,$$
$$\beta = 79^\circ\ 21',$$
$$\beta' = 59^\circ\ 43',$$

and if $\text{A} = 2$, $\text{A}' = 3$. The curve is AB in fig. 1, in which AC is the spiral for m, and AC′ that for m'.



But if $\text{A} = 2$, $\text{A}' = -3$, the curve is ABD, fig. 2, in which AC is the spiral for m, and AC′ for m', the cusp B corresponding to $\varphi = 5^\circ\ 26'$, and AB′D′ is the evolute.

Example 4. Suppose all the arbitrary constants to be zero, but those which correspond to a pair of imaginary roots of m. The term of φ is, in this case,

$$\text{B}e^{g\varphi} \sin (h\varphi + \varepsilon),$$

and if we suppose $\varepsilon = 0$, it is reduced to $\text{B}e^{g\varphi} \sin h\varphi$,

and if we take $\tan \gamma = \dfrac{h+1}{g}$, $\tan \gamma' = \dfrac{h-1}{g}$, $\text{B}' = \dfrac{\text{B}}{2g}$

we get $x = \text{B}'e^{g\varphi}[\cos \gamma \cos (\overline{h+1}.\varphi - \gamma) - \cos \gamma' \cos (\overline{h-1}.\varphi - \gamma')]$,

$$y = \text{B}'e^{g\varphi}[\cos \gamma \sin (\overline{h+1}.\varphi - \gamma) + \cos \gamma' \sin (\overline{h-1}.\varphi - \gamma')].$$

If we take the polar co-ordinates r and θ, θ being the angle which the radius vector r makes with the axis of x, we have to determine θ and r,

$$\tan (\theta + \tfrac{1}{2}\gamma - \tfrac{1}{2}\gamma' - \varphi) = \tan (\tfrac{1}{2}\gamma + \tfrac{1}{2}\gamma' - h\varphi).\cot \tfrac{1}{2}(\gamma' + \gamma) \cot \tfrac{1}{2}(\gamma - \gamma'),$$

$$r' = \frac{\text{B}'e^{g\varphi} \cos \gamma \sin(2h\varphi - \gamma - \gamma')}{\sin(\overline{h-1}.\varphi - \gamma' + \theta)}$$

$$= \text{B}'\varepsilon^{g\varphi} \sqrt{\cos^2 \gamma + \cos^2 \gamma' - 2\cos\gamma\cos\gamma'\cos(2h\varphi - \gamma + \gamma')}.$$

When $h\varphi = n\pi$, in which n is any integer, we have

$$r = \text{B}'e^{g\varphi} \sin (\gamma + \gamma'),$$

and θ such, that

$$\cos \theta = \cos n\pi \sin (\varphi - \gamma + \gamma') = \cos (\tfrac{3}{2}\pi - \gamma + \gamma' - \overline{h-1}\,.\,\varphi),$$
$$\sin \theta = -\cos n\pi \cos (\varphi - \gamma + \gamma') = \sin(\tfrac{3}{2}\pi - \gamma + \gamma' - \overline{h-1}\,.\,\varphi);$$

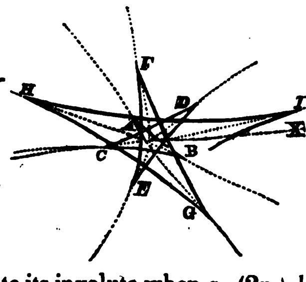
or $\qquad \theta = \tfrac{3}{2}\pi - \gamma + \gamma' - (h-1)\varphi = \tfrac{3}{2}\pi - \left(1 - \dfrac{1}{h}\right)n\pi - \gamma + \gamma'.$

Now at all such points the radius of curvature is zero, and the curve has a cusp of the first species; and each of these cusps is upon a logarithmic spiral whose equation is

$$r = \text{B}' \sin(\gamma + \gamma') e^{\frac{g}{h-1}(\frac{3}{2}\pi - \gamma + \gamma' - \theta)}$$

the radius vectors at the cusps making angles of $\left(1 - \frac{1}{h}\right)\pi$, with each other.

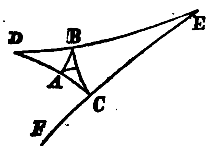
Thus in the figure, which is constructed for
$\text{A} = 254^\circ\ 21'$, $h = 6$, $g = 0{,}40417$,
$\gamma = 86^\circ\ 42'$, $\gamma' = 85^\circ\ 23'$, $\text{B}' = 1$;
the curve is, in its course, ABCDEFGHI, &c.



Example 5. The curve itself is parallel to its involute when $a = (2n + \frac{1}{2})\pi$ and if the co-efficient corresponding to $h = 1$ is zero, the curve is strictly its own involute without change of position in all other cases. Thus it is so for all the logarithmic spirals corresponding to the real root of the equation

$$e^{-m(2n - \frac{1}{2})\pi} - m = 0.$$

and the curve corresponding to a pair of imaginary roots would be of the form ABCDEFGHI, &c.



Example 6. A great variety of curves might be obtained by using several terms with various constants. Thus, if the real root of the equation

$$e^{-m(2n - \frac{1}{2})\pi} - m = 0$$

were combined with the root corresponding to the cycloid, a curve would be obtained which in the outset would hardly differ from the cycloid for very large negative values of φ, and it would approach very near the logarithmic spiral at the end where φ was very large.

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(95). QUESTION XIV. *By Prof. Avery.*

Suppose a rod to descend as in Question (63), Miscellany, and that a particle, whose weight is inconsiderable with respect to that of the rod, is placed on it and begins to descend by gravity, without friction, at the instant the rod commences its motion. Required the point on the rod where the particle must be placed, in order that it may arrive at the lowest extremity of the rod at the time the rod becomes horizontal.

FIRST SOLUTION. *By the Proposer.*

Let x, y be the co-ordinates of the place of the particle at any time t, from the origin of motion, a the length of the rod, φ the angle a line drawn from the origin to the centre of the rod makes with the axis of y,

vertical, r the distance of the particle from the foot of the rod; φ', r' the initial values of φ, r; then

$$x=(a-r)\sin\varphi,\quad y=r\cos\varphi\ .\ .\ .\ .\ .\ .\ (1),$$

and, by the solutions to question (63), Math. Miscellany,

$$dt=\sqrt{\frac{a}{3g}}\cdot\frac{d\varphi}{\sqrt{\cos\varphi'-\cos\varphi}}\ .\ .\ .\ .\ .\ .\ .\ (2).$$

By Dynamics,

$$\frac{d^2x}{dt^2}\delta x+\left(\frac{d^2y}{dt^2}+g\right)y=0\ .\ .\ .\ .\ .\ .\ (3);$$

substitute (1) and (2) in (3), r being the only variable with regard to δ,

$$\frac{d^2r}{dt^2}+(a\sin^2\varphi-r)\frac{d\varphi^2}{dt^2}-\tfrac{3}{2}g\sin^2\varphi\cos\varphi+g\cos\varphi=0\ .\ .\ (4).$$

Now, since, from (1), φ is a function of t, r is a function of t, and by Maclaurin's Theorem,

$$r=(r)+\left(\frac{dr}{dt}\right)t+\tfrac{1}{2}\left(\frac{d^2r}{dt^2}\right)t^2+\frac{1}{2\,.\,3}\left(\frac{d^3r}{dt^3}\right)t^3+\&c.\ .\ (5);$$

but, at the beginning of motion, when $t=0$, $\frac{d\varphi}{dt}=0$, $\frac{dr}{dt}=0$, by hypothesis, so that from (4), we easily get,

$$(r)=r',\left(\frac{dr}{dt}\right)=0,\left(\frac{d^2r}{dt^2}\right)=\tfrac{1}{2}g\cos\varphi'(1-3\cos^2\varphi'),\left(\frac{d^3r}{dt^3}\right)=0,\ \&c.,$$

Therefore $\qquad r=r'+\tfrac{1}{4}g\cos\varphi'(1-3\cos^2\varphi')t^2\ .\ .\ .\ .\ .\ .\ (6),$

and, by (2), when the rod is horizontal, and $r=0$,

$$r'=\tfrac{1}{4}g\cos\varphi'(3\cos^2\varphi'-1)t^2=\frac{a}{12}\cos\varphi'(3\cos^2\varphi'-1)\left\{\int_{\varphi'}^{\frac{1}{2}\pi}\frac{d\varphi}{\sqrt{\cos\varphi'-\cos\varphi}}\right\}^2(7),$$

the integral contained in this equation being easily determined from Legendre's Tables of Elliptic Functions.

Second Solution. *By Mr. O. Root.*

If R be the re-action of the rod, r the distance of the particle from the lowest point of the rod, and the notation otherwise as in my solution to question (63), the equations of its motion will be

$$\frac{d^2x}{dt^2}+\text{R}\cos\varphi=0,\quad\frac{d^2y}{dt^2}+\text{R}\sin\varphi+g=0\ .\ .\ .\ (1),$$

or, eliminating R, $\qquad -\frac{d^2x}{dt^2}\sin\varphi+\left(\frac{d^2y}{dt^2}+g\right)\cos\varphi=0\ .\ .\ (2).$

But $\qquad x=(2a-r)\sin\varphi,\quad y=r\cos\varphi\ .\ .\ .\ .\ .\ .\ (3)$

and (2) becomes, by substitution,

$$\frac{d^2r}{dt^2}+(2a\sin^2\varphi-r)\frac{d\varphi^2}{dt^2}-2a\sin\varphi\cos\varphi\,\frac{d^2\varphi}{dt^2}+g\cos\varphi=0\ .\ (4),$$

and, by Maclaurin's Theorem,

$$r=r'+\left(\frac{dr'}{dt}\right)t+\tfrac{1}{2}\,.\left(\frac{d^2r'}{dt^2}\right)t^2+\&c.\ .\ .\ .\ (5),$$

therefore if T represent the whole time of motion, or

$$\text{T} = \sqrt{\frac{2a}{3g}} \cdot \int_{\varphi'}^{\frac{1}{2}\pi} \frac{d\varphi}{\sqrt{\cos\varphi' - \cos\varphi}} \quad . \quad . \quad . \quad . \quad . \quad (6),$$

we have, at the end of the motion, since $\left(\frac{dr'}{dt}\right) = 0$,

$$r' = -\tfrac{1}{2}\left(\frac{d^2 r'}{dt^2}\right)\text{T}^2 - \frac{1}{2\,.\,3}\left(\frac{d^3 r'}{dt^3}\right)\text{T}^3 - \&\text{c.} \quad . \quad . \quad (7),$$

the values of $\left(\frac{d^2 r'}{dt^2}\right)$, $\left(\frac{d^3 r'}{dt^3}\right)$, &c., being derived from (4), when $t = 0$, and $\varphi = \varphi'$; and the values of $\left(\frac{d\varphi}{dt}\right)$, $\left(\frac{d^2\varphi}{dt^2}\right)$, &c., from their values at page 312, vol. I.

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(96). QUESTION XV. *By Mr. W. S. B. Woolhouse, London.*

A crown piece being twirled any how on a perfectly smooth horizontal plane, it is required to investigate the circumstances of the motion and the velocities of its points when it acquires any given position, disregarding the thickness of the metal.

FIRST SOLUTION. *By Dr. Strong.*

Let the notation be as in the article on Rotary motion, in a subsequent part of this Number of the Miscellany; the axes of x and y being fixed on the horizontal plane, that of z vertical and counted downwards; X, Y, Z will be the co-ordinates of the centre of the plate; the co-ordinates x, y, z of any element dm of the plane having their origin at that point. Then, if we *fix* the axes of x', y', z' in the plate, that of z' being its axis, and those of x', y' on its plane, the angle φ of that article will represent the angle which the axis of x' makes with the tangent, at the point of contact of the plate and plane, to the path of that point; χ will be the angle made by the axis of x, with the same tangent, and θ will be the inclination of the plate to the horizon. Then if these co-ordinates refer to the point of contact, x_1, y_1 being the co-ordinates of the same point referred to the first axes, ds the element of the path of the point on the plane, R_1 the radius of the plate, and M' its mass, we shall have for this point,

$$\left.\begin{array}{l} x' = \text{R}_1 \sin\varphi,\ y' = \text{R}_1 \cos\varphi,\ z' = 0; \\ x_1 - \text{X} = x = ax' + by' = \text{R}_1 \cos\theta \sin\chi, \\ y_1 - \text{Y} = y = a'x' + b'y' = \text{R}_1 \cos\theta \sin\chi, \\ \therefore\ \text{Z} = \sqrt{\text{R}_1{}^2 - (x_1 - \text{X})^2 - (y_1 - \text{Y})^2} = \text{R}_1 \sin\theta \end{array}\right\} \quad . \quad . \quad . \quad . \quad (\text{A}),$$

For a slight change of position, when x_1, y_1 become $x_1 + dx_1, y_1 + dy_1$, we have

$$(x_1 - \text{X})dx_1 + (y_1 - \text{Y})dy_1 = 0, \text{ or } \sin\chi dx_1 + \cos\chi dy_1 = 0;$$

which is satisfied by putting $\frac{dx_1}{ds} = \cos\chi$ $\frac{dy_1}{ds} = \sin\chi$. Now let $\text{M}'\text{T} =$ the resistance the plate meets with in the direction of the element ds

arising from friction; $M'V$ = the resistance in the plane in a direction at right angles to ds, and $M'R''$ = the re-action of the plane in a vertical direction. Then the resultants of all the accelerating forces acting on any element, dm, in the direction of the three axes, being represented by X', Y', Z', are

$$\left.\begin{aligned} X' &= -T\frac{dx_1}{ds} + V\frac{dy_1}{ds} = -T\cos\chi - V\sin\chi, \\ Y' &= -T\frac{dy_1}{ds} - V\frac{dx_1}{ds} = T\sin\chi - V\cos\chi, \end{aligned}\right\} \quad . \quad (B).$$
$$Z' = R'' - g$$

the equations of motion (b') of the centre of gravity, become

$$\frac{d^2X}{dt^2} = -T\cos\chi - V\sin\chi,\ \frac{d^2Y}{dt^2} = T\sin\chi - V\cos\chi,\ \frac{d^2Z}{dt^2} = R'' - g \quad . \quad (C).$$

$$\left.\begin{aligned} \therefore\ T &= -\frac{d^2X}{dt^2}\cos\chi + \frac{d^2Y}{dt^2}\sin\chi, \\ V &= -\frac{d^2X}{dt^2}\sin\chi - \frac{d^2Y}{dt^2}\cos\chi, \\ R'' &= \frac{d^2Z}{dt^2} + g \end{aligned}\right\} \quad . \quad . \quad (D).$$

The equations of rotation of the plate around the axes of x', y', z', are those in (z') of the article cited, which are easily adapted to this case. For since $z' = 0$, and the plate is symmetrical and homogeneous, by (k),

$$A = B = Sx'^2 dm = \tfrac{1}{4}R_1{}^2M',\ C = 2A = \tfrac{1}{2}R_1{}^2M';$$
$$P = -T\cos\chi - V\sin\chi,\ Q = T\sin\chi - V\cos\chi,\ R = R'';$$

by substituting in (n) the proper values of a, b, c, &c., given in (10),

$$R' = R''\cos\theta - V\sin\theta,\ Q' = T\sin\varphi - V\cos\varphi\cos\theta - R''\sin\theta\cos\varphi,$$

$P' = -T\cos\varphi - V\cos\theta\sin\varphi - R''\sin\theta\sin\varphi$, and the equations ($z'$) become

$$\left.\begin{aligned} \frac{dp}{dt} + qr &= \frac{4\cos\varphi}{R_1}(R''\cos\theta - V\sin\theta,) \\ \frac{dq}{dt} - pr &= -\frac{4\sin\varphi}{R_1}(R''\cos\theta - V\sin\theta, \\ \frac{dr}{dt} &= \frac{2T}{R_1'^1} \end{aligned}\right\} \quad . \quad . \quad . \quad (E),$$

In the case where the plate rolls, without sliding, the point of contact may be regarded as momentarily at rest in consequence of the opposite motions arising from the motion of the centre of gravity, and of rotation; then

$$\frac{dx_1}{dt} = \frac{dX}{dt} + \frac{dx}{dt} = 0,\ \frac{dy_1}{dt} = \frac{dY}{dt} + \frac{dy}{dt} = 0 \quad . \quad . \quad . \quad (F)$$

But equations (f) become in this case,

$L = -ry' = -R_1 r\cos\varphi$, $M = rx' = R_1 r\sin\varphi$, $N = py' - qx' = R_1(p\cos\varphi - q\sin\varphi)$, and substituting these, together with the values of a, b, c, &c., from (10), in (h),

$$\left.\begin{aligned}\frac{d\mathrm{x}}{dt}=-\frac{dx}{dt}=\mathrm{R}_1\left(r\cos\chi+\sin\theta\sin\chi\frac{d\theta}{dt}\right),\\ \frac{d\mathrm{Y}}{dt}=-\frac{dy}{dt}=\mathrm{R}_1\left(-r\sin\chi+\sin\theta\cos\chi\frac{d\theta}{dt}\right),\end{aligned}\right\}\quad\ldots\quad(\mathrm{G});$$

therefore, (D) become, by substituting these

$$\mathrm{V}=\mathrm{R}_1\left(r\frac{d\chi}{dt}+\frac{d^2\cos\theta}{dt^2}\right),\ \mathrm{T}=-\mathrm{R}_1\left(\frac{dr}{dt}+\frac{d\theta}{dt}\cdot\frac{d\chi}{dt}\sin\theta\right),$$
$$\mathrm{R}''=\mathrm{R}_1\frac{d^2.\sin\theta}{dt^2}+g.$$

and these substituted in (E), together with the value of p, q, r, from (e), give the equations for rolling motion,

$$\left.\begin{aligned}3\frac{dr}{dt}+2\sin\theta\frac{d\theta}{dt}\cdot\frac{d\chi}{dt}=0,\\ d\left(\sin^2\theta\frac{d\chi}{dt}\right)-2rd.\cos\theta=0,\\ \tfrac{5}{4}\cdot\frac{d^2\theta}{dt^2}-\tfrac{1}{4}\sin\theta\left(\cos\theta\frac{d\chi}{dt}+6p\right)\frac{d\chi}{dt}+\frac{g}{\mathrm{R}_1}\cos\theta=0,\end{aligned}\right\}\quad.\quad(\mathrm{H}).$$

which agree with the equations found at p. 66, No. 10, of the Math. Diary.

Again, if the plane be perfectly smooth, so that there is no friction, $\mathrm{T}=0$, $\mathrm{V}=0$, $\mathrm{R}''=\mathrm{R}_1\dfrac{d^2.\sin\theta}{dt^2}+g$, and (C) become $\dfrac{d^2\mathrm{X}}{dt^2}=0$, $\dfrac{d^2\mathrm{Y}}{dt^2}=0$, or integrating

$$\mathrm{X}=\mathrm{A}t+\mathrm{B},\ \mathrm{Y}=\mathrm{A}'t+\mathrm{B}',$$

A, B, A$'$, B$'$ being constants, determined from the initial position and velocity of the centre, and eliminating t we get $\mathrm{A}'\mathrm{X}-\mathrm{A}\mathrm{Y}=\mathrm{A}'\mathrm{B}-\mathrm{A}\mathrm{B}'$, which shows that the centre of the plate is always on the same vertical plane, having its height from the horzontal plane $z=\mathrm{R}_1\sin\theta$. The equations (E) of rotation become by substituting for p, q, r and slightly modifying them

$$\left.\begin{aligned}-\frac{d^2\theta}{dt^2}+\frac{d\chi^2}{dt^2}\sin\theta\cos\theta+2r\sin\theta\frac{d\chi}{dt}=4\cos\theta\left(\frac{d^2\sin\theta}{dt^2}+\frac{g}{\mathrm{R}_1}\right),\\ d\left(\sin^2\theta\frac{d\chi}{dt}\right)+2r\sin\theta d\theta=0,\\ dr=0.\end{aligned}\right\}\quad.\quad(\mathrm{I}).$$

The last of these gives $r=$ const. $=$ the angular velocity of the plate about its axis, then the second is integrable and gives

$$\sin^2\theta\,.\frac{d\chi}{dt}-2r\cos\theta=\text{const.}=n\quad.\ .\ .\ .\ .\ .\quad(\mathrm{K}).$$

If the first be multiplied by $d\theta$, and subtracted from the second multiplied by $d\chi$, the integral of the resulting equation is

$$\left.\begin{aligned}\frac{d\theta^2}{dt^2}+4\frac{(d\sin\theta)^2}{dt^2}+\sin^2\theta\frac{d\chi^2}{dt^2}+\frac{8g}{\mathrm{R}_1}\sin\theta=\text{const.}=m,\\ \text{or, by (K)},\ (\sin^2\theta+\sin^2 2\theta)\frac{d\theta^2}{dt^2}=m\sin^2\theta-(2r\cos\theta+n)^2-\frac{8g}{\mathrm{R}_1}\sin^3\theta\end{aligned}\right\}(\mathrm{L});$$

$$\left.\begin{aligned} &\text{and, by } (e), && \frac{d\varphi}{dt} - \cos\theta\frac{d\chi}{dt} = r, \\ &\text{or} && \sin^2\theta\frac{d\varphi}{dt} = r(1+\cos^2\theta) + n\cos\theta, \end{aligned}\right\} \quad \ldots\ldots \quad (\text{M}).$$

The angular motions are therefore all given in functions of θ by (K), (L), (M). If we suppose $\frac{d\chi}{dt} = \text{const.}$, (K) will give $\theta = \text{const.}$, (M) will give $\frac{d\varphi}{dt} = \text{const.}$, therefore the plate would revolve uniformly about its axis, the point of contact describing a circle with a uniform motion on the horizontal plane, when $\frac{d\text{X}}{dt} = 0$, $\frac{d\text{Y}}{dt} = 0$; but if these are not $= 0$, the aforesaid motion continues, while the centre of the plate describes a straight line parallel to the horizontal plane. If $\frac{d\chi}{dt} = 0$, the point of contact describes a right line on the plane, parallel to the right line described by the centre, and the plate revolves uniformly about its axis in all cases, except when $r = 0$.

SECOND SOLUTION. *By Prof. C. Avery.*

Let X, Y, Z be the co-ordinates of any element dm of the crown-piece, referred to three rectangular axes, the two first on the horizontal plane on which the disc moves; t the time from any epoch; φ, χ, θ the three angles which determine the position of the crown-piece and of its principal axes at any time; R the radius of the disc, and m its mass. Then we have by the general formula of Dynamics

$$\text{S}dm\left(\frac{d^2\text{X}}{dt^2}\delta\text{X} + \frac{d^2\text{Y}}{dt^2}\delta\text{Y} + \frac{d^2\text{Z}}{dt^2}\delta\text{Z} + g\delta\text{Z}\right) = 0 \ . \quad . \ (1).$$

Assume

$$\begin{aligned} \text{X} &= x + ax' + by', \\ \text{Y} &= y + a'x' + b'y', \\ \text{Z} &= z + a''x' + b''y', \end{aligned}$$

where x, y, z are the co-ordinates of the centre of the plate, referred to the same axes; x', y', z' the co-ordinates of dm, referred to three rectangular axes passing through the centre of gravity, those of x', y', in the disc itself, and that of z' perpendicular to it, z' being neglected on account of its minuteness; a, b, a', &c., having the values in equations (10) of Dr. Strong's Article on Rotation in this number.*

The crown-piece will be subjected to move on the horizontal plane by the equation of condition

$$\text{L} = z - \text{R}\sin\theta = 0 \ . \ . \ . \ . \ . \ . \ . \ . \ (2).$$

and if we put, to facilitate the transformation of (1),

$$2\text{T} = \text{S}dm\left(\frac{d\text{X}^2}{dt^2} + \frac{d\text{Y}^2}{dt^2} + \frac{d\text{Z}^2}{dt^2}\right), \text{V} = g\text{S}dmz \ . \ . \ . \ (3);$$

then, since

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* The Editor has taken the liberty of changing these references for the reader's convenience, so as to prevent the necessity of his consulting a number of different authors.

$Sdm = m$, $Sx'dm = Sy'dm = 0$, $Sx'^2 dm = Sy'^2 dm = \frac{1}{4}R^2 m$, $Sx'y'dm = 0$,

$$2T = m \cdot \frac{dx^2 + dy^2 + dz^2}{dt^2} + \frac{1}{4}R^2 m \cdot \frac{da^2 + da'^2 + da''^2 + db^2 + db'^2 + db''^2}{dt^2},$$

$$V = gmz.$$

But, using Dr. Strong's equations of reduction, in (c') and (e),

$$da^2 + da'^2 + da''^2 + db^2 + db'^2 + db''^2 = (p^2 + q^2 + 2r^2)dt^2$$
$$= d\theta^2 + (1 + \cos^2\theta)d\chi^2 + 2d\varphi^2 - 4\cos\theta d\varphi d\chi$$

$$\therefore 2T = m \frac{dx^2 + dy^2 + dz^2}{dt^2} + \frac{R^2 m}{4}\left\{ \frac{d\theta^2}{dt^2} + (1 + \cos^2\theta)\frac{d\chi^2}{dt^2} + 2\frac{d\varphi^2}{dt^2} - 4\cos\theta \frac{d\varphi}{dt} \cdot \frac{d\chi}{dt} \right\}.$$

Then there will result, as La Grange has shown, from the substitution in (1), for each of the six quantities x, y, z, θ, χ, φ that enter into it, an equation of the form

$$d \cdot \frac{\delta T}{\delta dx} - \frac{\delta T}{\delta x} + \frac{\delta V}{\delta x} + \lambda \cdot \frac{\delta L}{\delta x} = 0,$$

λ being an indeterminate co-efficient. Hence the equations of motion

$$\left.\begin{aligned} \frac{d^2 x}{dt^2} = 0, \quad \frac{d^2 y}{dt^2} = 0, \quad \frac{d^2 z}{dt^2} + g + \frac{\lambda}{m} = 0, \\ \frac{d^2\theta}{dt^2} + \tfrac{1}{2}\sin 2\theta \frac{d\chi^2}{dt^2} - 2\sin\theta \frac{d\varphi}{dt} \cdot \frac{d\chi}{dt} - \frac{4\lambda}{Rm}\cos\theta = 0, \\ \frac{d \cdot \{(1 + \cos^2\theta)d\chi - 2\cos\theta d\varphi\}}{dt^2} = 0, \\ \frac{d \cdot \{d\varphi - \cos\theta d\chi\}}{dt^2} = 0, \end{aligned}\right\} \quad . \quad (4).$$

The two first equations give, by integration,

$$x = a_1 t + b_1, \quad y = a_2 t + b_2 \quad . \quad . \quad . \quad . \quad . \quad (5)$$

which show that the centre of the disc is confined to a vertical plane, the position of which depends on the arbitrary constants a_1, a_2, b_1, b_2, or on the initial position and impulse of that point. The third give

$$\frac{\lambda}{m} = -\frac{d^2 z}{dt^2} - g = -R\frac{d^2(\sin\theta)}{dt^2} - g \cdot \quad . \quad . \quad . \quad (6),$$

and the fifth and sixth,

$$\left.\begin{aligned} (1 + \cos^2\theta)\frac{d\chi}{dt} - 2\cos\theta\frac{d\varphi}{dt} = \text{constant} = n, \\ \frac{d\varphi}{dt} - \cos\theta\frac{d\chi}{dt} = \text{constant} = r, \end{aligned}\right\} \quad . \quad . \quad . \quad (7),$$

or

$$\left.\begin{aligned} \sin^2\theta \cdot \frac{d\chi}{dt} = 2r\cos\theta + n, \\ \sin^2\theta \cdot \frac{d\varphi}{dt} = (1 + \cos^2\theta)r + n\cos\theta, \end{aligned}\right\} \quad . \quad . \quad . \quad (8).$$

For another integral, we have in this case, the equation of living forces,

$$T + V = \text{const.}.$$

or combining the constants $\frac{dx}{dt}$, $\frac{dy}{dt}$ with this constant, and reducing

$$(1+4\cos^2\theta)\frac{d\theta^2}{dt^2}+(1+\cos^2\theta)\frac{d\chi^2}{dt^2}+2\frac{d\varphi^2}{dt^2}-4\cos\theta\frac{d\varphi}{dt}.\frac{d\chi}{dt}+\frac{8g}{\text{R}}\sin\theta=m,$$

or, by (8),

$$(\sin^2\theta+\sin^2 2\theta)\frac{d\theta^2}{dt^2}+(2r\cos\theta+n)^2+(2r^2-m)\sin^2\theta+\frac{8g}{\text{R}}\sin^3\theta=0 \quad (9).$$

The equations (8) and (9) give the angular velocities, at any time, in terms of θ, and the constants, which are to be determined from the initial impulses. Since r represents the velocity of rotation about the axis of the disc, that velocity is constant. Equation (6) shows the vertical pressure, p, of the disc on the plane to be, at the time t,

$$p=gm+\text{R}m.\frac{d^2(\sin\theta)}{dt^2} \quad \ldots\ldots\ldots \quad (10).$$

The path of the centre is determined from the equations (2) and (5), and that of the point of contact on the plane, by making $x'=\text{R}\sin\varphi$, $y'=\text{R}\cos\varphi$; then its co-ordinates are, $\text{z}=0$,

$$\text{X}=x+a\,x'+b\,y'=x+\text{R}\cos\theta\sin\chi,$$
$$\text{Y}=y+a'x'+b'y'=y+\text{R}\cos\theta\cos\chi,$$

between which, x, y, θ, χ may be eliminated since they are all functions of t.

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(97). Question XVI. *By Mr. W. S. B. Woolhouse*

It is required to solve the preceding Question, when, instead of the circular disc, any solid of revolution is substituted, as for instance, a *spheroid*, the semi-axes of which are a and b.

First Solution. *By Dr. Strong.*

Let the centre of gravity of the *ellipsoid* be defined at any time t, by the rectangular co-ordinates X, Y, Z the first two being in the horizontal plane and the last vertical and directed upwards, and let the co-ordinates of any element dm of the body be x, y, z when referred to these axes, and x', y', z' when referred to the principal axes A', B', C' of the ellipsoid as axes of co-ordinates, then the equation of the ellipsoid (when the point x', y', z' is at the surface) is

$$u=\text{A}'^2\text{B}'^2z'^2+\text{A}'^2\text{C}'^2y'^2+\text{B}'^2\text{C}'^2x'^2-\text{A}'^2\text{B}'^2\text{C}'^2=0 \quad . \quad . \ (\text{A}),$$

but from the eq. of transformation (3), (see the article on Rotary Motion),

$$x'=a(x-\text{X})+a'(y-\text{Y})+a''(z-\text{Z}),$$
$$y'=b(x-\text{X})+b'(y-\text{Y})+b''(z-\text{Z}),$$
$$z'=c(x-\text{X})+c'(y-\text{Y})+c''(z-\text{Z}),$$

and if these be substituted in (A) we shall have the equation of the surface in function of the fixed co-ordinates x, y, z. Hence if we suppose two sections of the solid, passing through the point of contact of the body with the plane, and parallel to the co-ordinates planes xz, yz respectively, we shall have

$$\frac{du}{dx}dx+\frac{du}{dz}dz=0,\ \frac{du}{dy}dy+\frac{du}{dz}dz=0;$$

which will enable us to draw a tangent at any point of either section, but the horizontal plane touches each section, and for the point of contact

$$z=0, dz=0, \frac{du}{dx}=0, \frac{du}{dy}=0.$$

But by the above equations,

$$\frac{du}{dx}=\frac{du}{dx'}.\frac{dx'}{dx}+\frac{du}{dy'}.\frac{dy'}{dx}+\frac{du}{dz'}.\frac{dz'}{dx}=\text{B}'^2\text{C}'^2 a\,x'+\text{A}'^2\text{C}'^2 b\,y'+\text{A}'^2\text{B}'^2 c\,z'=0,$$

$$\frac{du}{dy}=\frac{du}{dx'}.\frac{dx'}{dy}+\frac{du}{dy'}.\frac{dy'}{dy}+\frac{du}{dz'}.\frac{dz'}{dy}=\text{B}'^2\text{C}'^2 a'x'+\text{A}'^2\text{C}'^2 b'y'+\text{A}'^2\text{B}'^2 c'z'=0;$$

and it is evident, from eq. (4) of transformation, that these equations, as well as (A,) will be satisfied by putting

$$x'=\text{A}'^2 a''v, y'=\text{B}'^2 b''v, z'=\text{C}'^2 c''v, \text{ if } v=\pm(\text{A}'^2a''^2+\text{B}'^2b''^2+\text{C}'^2c''^2)^{-\frac{1}{2}} \text{ (B)}.$$

Also, since $z - \text{z} = a''x' + b''y' + c''z'$, for this point, where $z=0$,

$$\text{z} = -(a''x'+b''y'+c''z') = \pm\frac{1}{v} = (\text{A}'^2a''^2+\text{B}'^2b''^2+\text{C}'^2c''^2)^{\frac{1}{2}} \text{ . (C)}.$$

Let M′ be the mass of the plate, and M′R″ the vertical reaction of the plane upon the solid; then the equations of translation of the centre are

$$\frac{d^2\text{X}}{dt^2}=0, \frac{d^2\text{Y}}{dt^2}=0, \frac{d^2\text{Z}}{dt^2}=\text{R}''-g \text{ . . . . (D)},$$

therefore, $\quad \text{X}=\text{A}_1t+\text{A}_2,\ \text{Y}=\text{B}_1t+\text{B}_2,\ \text{R}''=\frac{d^2\text{Z}}{dt}+g.$

$\text{A}_1, \text{A}_2, \text{B}_1, \text{B}_2$ being arbitrary constants; eliminating t, we have

$$\text{A}_1\text{Y}-\text{B}_1\text{X}=\text{A}_1\text{B}_2-\text{A}_2\text{B}_1,$$

which shows that the centre of the ellipsoid is always in the same vertical plane. The equations of rotation are those in (z') of the subsequent article on rotation, and are adopted to this case by making

$$\text{P}=0, \text{Q}=0, \text{P}'=a''\text{R}, \text{Q}'=b''\text{R}, \text{R}'=c''\text{R}, \text{S}dm\text{R}=\text{M}'\text{R}'',$$

and writing in them the values of x', y', z' found in (B), after putting $v=\frac{1}{\text{z}}$; they thus become

$$\left.\begin{aligned}\text{A}\frac{dp}{dt}+(\text{C}-\text{B})qr&=\frac{\text{M}'\text{R}''}{\text{z}}(\text{C}'^2-\text{B}'^2)b''c'',\\ \text{B}\frac{dq}{dt}+(\text{A}-\text{C})pr&=\frac{\text{M}'\text{R}''}{\text{z}}(\text{A}'^2-\text{C}'^2)a''c'',\\ \text{C}\frac{dr}{dt}+(\text{B}-\text{A})pq&=\frac{\text{M}'\text{R}''}{\text{z}}(\text{B}'^2-\text{A}'^2)a''b'',\end{aligned}\right\} \text{ . . . (E)}.$$

If we multiply these equations severally by a'', b'', c'' and add them together, reduce the result by equations (c') of the article on Rotation, we get

$$d.[\text{A}a''p+\text{B}b''q+\text{C}c''r]=0,$$

and $\quad \text{A}a''p+\text{B}b''q+\text{C}c''r = \text{const.} = m' \text{ . . (F)}$

Multiply (E) by pdt, qdt, rdt, severally, add the products, and reduce by the same equations and by (C), then

$$\text{A}pdp+\text{B}qdq+\text{C}rdr=-\text{M}'\text{R}''dz=-\text{M}'dz\left(\frac{d^2\text{z}}{dt^2}+g\right),$$

or, integrating,

$$Ap^2 + Bq^2 + Cr^2 + M'\left(\frac{dz^2}{dt^2} + 2gz\right) = \text{const.} = n' \quad . \quad (G).$$

See Poisson's Traité de Mecanique, Vol. II., page 178, ed. 1811, from which the above solution, with some slight variations, has been taken.

If we put $A' = B'$, the solid becomes a spheroid, the axis of z' being the axis of revolution, and those of x', y' two rectangular diameters of its equator, then

$$A = B = S(x'^2 + z'^2)\,dm = \tfrac{1}{5}M'(A'^2 + C'^2),\ C = S(x'^2 + y'^2)dm = \tfrac{2}{5}M'A'^2,$$

$$z = \sqrt{A'^2(a''^2 + b''^2) + C'^2c''^2} = \sqrt{A'^2\sin^2\theta + C'^2\cos^2\theta},$$

and put $m' = \frac{1}{5}M'l$, $n' = \frac{1}{5}M'l'$; then the last of equations (E) together with (F) and (G) become

$$\left.\begin{aligned} &dr = 0, \text{ or } r = \text{constant},\\ &(A'^2+C'^2)(a''p + b''q) + 2A'^2c''r = l,\\ &(A'^2+C'^2)(p^2+q^2) + 2A'^2r + 5\left(\frac{dz^2}{dt^2} + 2gz\right) = l', \end{aligned}\right\} \quad . \quad . \quad . \quad (H).$$

Write in these the values of p, q, r from equations (e), and those of a'', b'', c'' from equations (10) of the subsequent article, restoring the value of z, and putting

$$\frac{2rA'^2}{A'^2 + C'^2} = E,\ \frac{5(A'^2 - C'^2)^2}{A'^2 + C'^2} = G^2,\ \frac{10g}{A'^2 + C'^2} = g',$$

$$\frac{l}{A'^2 + C'^2} = -F,\ \frac{l'}{A'^2 + C'^2} = H,$$

$$\left.\begin{aligned} &\frac{d\varphi}{dt} - \cos\theta\frac{d\chi}{dt} = r,\\ &\sin^2\theta\frac{d\chi}{dt} - E\cos\theta = F,\\ &\frac{d\theta^2}{dt^2} + \sin^2\theta\frac{d\chi^2}{dt^2} + \frac{G^2\sin^2\theta\cos^2\theta}{A'^2\sin^2\theta + C'^2\cos^2\theta}.\frac{d\theta^2}{dt^2} + g'\sqrt{A'^2\sin^2\theta + C'^2\cos^2\theta} = H, \end{aligned}\right\}(I)$$

the constants r, F, H being determined by any given state of the body. These equations give the three angular velocities of the body in terms of θ; and by eliminating χ from the second and third, θ can be found from the resulting equation in terms of t.

If C' be so small that it may be neglected, and $A' = R_1$, these equations become

$$\left.\begin{aligned} &\frac{d\varphi}{dt} - \cos\theta\frac{d\chi}{dt} = r,\\ &\sin^2\theta\frac{d\chi}{dt} - 2r\cos\theta = F,\\ &\frac{d\theta^2}{dt^2} + \sin^2\theta\frac{d\chi^2}{dt^2} + 5\cos^2\theta.\frac{d\theta^2}{dt^2} + \frac{10g}{R_1}.\sin\theta = H, \end{aligned}\right\} \quad . \quad . \quad . \quad (K).$$

By comparing these with the corresponding ones (K), (L), (M) in the solution to the last question, we perceive that the two first agree with (K) and (M), but the last does not agree with (L), the quantity $4\frac{(d\sin\theta)^2}{dt^2} + \frac{8g}{R_1}\sin\theta$, in (L), corresponding to the terms $5\frac{(d\sin\theta)^2}{dt^2} + \frac{10g}{R_1}\sin\theta$

in this equation; hence the motion of a very thin spheroidal disk is not the same as that of a very thin cylindrical disk; and the reason is that a greater portion of the mass is accumulated round the centre of the one than the other, ***since, however thin, a spheroidal disk cannot be considered as equally thick throughout.***

SECOND SOLUTION. *By Prof. M. Catlin, Hamilton College.*

Let x, y, and z be the rectangular co-ordinates of the centre of gravity, the plane xy being horizontal; R the reaction of the plane, and M the mass of the moving body. Then we shall have for the motion of the centre of gravity,

$$\text{M}\frac{d^2x}{dt^2}=0,\ \text{M}\frac{d^2y}{dt^2}=0,\ \text{M}\left(\frac{d^2z}{dt^2}+g\right)-\text{R}=0 \quad . \quad . \quad (1).$$

Let x', y', z' be the point of contact of the solid with the horizontal plane, referred to the principal axes passing through the centre of gravity; A, B, C the moments of inertia, and p, q, r the angular velocities about the same axes. Then, p, q, r being as in equation (e), and a'', b'', c'' as in eq. (10), of Dr. Strong's Article on Rotation, we shall have, for the rotatory motion of the body,

$$\left.\begin{aligned}\text{A}dp+(\text{C}-\text{B})qrdt&=\text{M}\left(\frac{d^2z}{dt^2}+g\right)(c''y'-b''z')dt\\ \text{B}dq+(\text{A}-\text{C})rpdt&=\text{M}\left(\frac{d^2z}{dt^2}+g\right)(a''z'-c''x')dt\\ \text{C}dr+(\text{B}-\text{A})pqdt&=\text{M}\left(\frac{d^2z}{dt^2}+g\right)(b''x'-a''y')dt\end{aligned}\right\}\quad(2).$$

Let $\text{L}=0$ be the equation of the surface of the moveable referred to the axes of x', y' and z', and put

$$\text{V}=\left[\left(\frac{d\text{L}}{dx'}\right)^2+\left(\frac{d\text{L}}{dy'}\right)^2+\left(\frac{d\text{L}}{dz'}\right)^2\right]^{-\frac{1}{2}}$$

then the above general equations must be subjected to the following conditions, which are equivalent to four independent equations, $\text{L}=0$,

$$z+a''x'+b''y'+c''z'=0,\ a''=\text{V}\frac{d\text{L}}{dx'},\ b''=\text{V}\frac{d\text{L}}{dy'},\ c''=\text{V}\frac{d\text{L}}{dz'} \quad . \quad . \quad (3).$$

The equations above are sufficient to completely determine the motion of any solid, twirled in any manner upon a smooth horizontal plane. We shall now proceed, as required by the question, to investigate the particular case of a solid of revolution.

Let the axis of z' be the axis of the figure—then evidently we shall have $x'=y'\tan\varphi$, but $a''=b''\tan\varphi$; hence $b''x'-a''y'=0$. We have also $\text{A}=\text{B}$, consequently equations (2) become by substitution

$$\left.\begin{aligned}\text{A}dp+(\text{C}-\text{A})qrdt&=\text{M}\left(\frac{d^2z}{dt^2}+g\right)(c''y'-b''z')dt\\ \text{A}dq+(\text{A}-\text{C})rpdt&=\text{M}\left(\frac{d^2z}{dt^2}+g\right)(a''z'-c''x')dt\\ \text{C}dr&=0\end{aligned}\right\}\quad(4).$$

Integrating (4) by the the usual method we obtain

$$\left.\begin{array}{l}\text{A}(p^2+q^2)+\text{M}\left(\frac{dz^2}{dt^2}+2gz\right)=k\\ \text{A}(a''p+b''q)+\text{C}c''r=k',\ r=k''\end{array}\right\}\quad .\ .\ .\ .\ (5),$$

where k, k' and k'' are arbitrary constants. The first k includes $\text{C}k''$.

But $\quad a''p+b''q=-\sin^2\theta\frac{d\chi}{dt};\ p^2+q^2=\sin^2\theta\frac{d\chi^2}{dt^2}+\frac{d\theta^2}{dt^2};$

therefore equations (5) are reduced to

$$\left.\begin{array}{l}\frac{d\chi}{dt}=\frac{\text{C}k''\cos\theta-k'}{\text{A}\sin^2\theta},\ \frac{d\varphi}{dt}=\frac{\cos\theta(\text{C}k''\cos\theta-k')+\text{A}k''\sin^2\theta}{\text{A}\sin^2\theta},\\ \text{A}\frac{d\theta^2}{dt^2}+\text{M}\frac{dz^2}{dt^2}=\frac{\text{A}\sin^2\theta(k-2g\text{M}z)-(\text{C}k''\cos\theta-k')^2}{\text{A}\sin^2\theta}\end{array}\right\}\ (6).$$

When the form of the equation of $\text{L}=0$ is given, z and $\frac{dz}{dt}$ will be known by means of (3) in terms of θ, φ and χ, which being substituted in (6) will give the velocities $\frac{d\chi}{dt}$, $\frac{d\varphi}{dt}$ and $\frac{d\theta}{dt}$ for any given position of the moveable. Integrating (1)

$$\frac{dx}{dt}=l,\ \frac{dy}{dt}=l',\ x=lt+m,\ y=l't+m'\quad .\ .\ .\quad (7).$$

Equations (7) determine the velocity of the centre of gravity, which is uniform in a direction parallel to the horizontal plane, and the projection of the locus of the centre of gravity upon the plane (xy) is a right line.

By integrating the third of (6) we may find the time t in terms of θ, which being substituted in (12), will give us the values of x, y, and z for any given values of θ, φ and χ. The locus of the point of contact will then become known by means of the equations

$$x_1=x+ax'+by'+cz',\ y_1=y+a'x'+b'y'+c'z'\ .\ \ .\ (8),$$

where x_1 and y_1 are the co-ordinates of that point referred to the axes of x and y, and a, b, &c., are as in Dr. Strong's eqs. (10).

We will now proceed to examine the less general case of a spheroid of revolution. We shall now have

$$\text{L}=\beta^2z'^2+\alpha^2(x'^2+y'^2)-\alpha^2\beta^2\quad .\ .\ .\ .\quad (9),$$

α and β representing the semi-axes of the ellipsoid. Taking the partial differentials of (9), with respect to x', y', z', and substituting them, together with the values of a'', b'', c'' in (3), we have after reduction

$$\left.\begin{array}{l}z=\sqrt{\alpha^2\cos^2\theta+\beta^2\sin^2\theta},\ \frac{dz}{dt}=\frac{(\beta^2-\alpha^2)\sin\theta\cos\theta}{z}\cdot\frac{d\theta}{dt},\\ x'=\frac{\beta^2\sin\theta\sin\varphi}{z},\ y'=\frac{\beta^2\sin\theta\cos\varphi}{z},\ z'=\frac{\alpha^2\cos\theta}{z}\end{array}\right\}\ .\ (10),$$

and equations (6) become, for the spheroid of revolution,

$$\left.\begin{array}{l}\frac{d\chi}{dt}=\frac{\text{C}k''\cos\theta-k'}{\text{A}\sin^2\theta},\ \frac{d\varphi}{dt}=\frac{\cos\theta(\text{C}k''\cos\theta-k')+\text{A}k''\sin^2\theta}{\text{A}\sin^2\theta},\\ \frac{d\theta}{dt}=\frac{z}{\sin\theta}\sqrt{\frac{\text{A}\sin^2\theta(k-2g\text{M}z)-(\text{C}k''\cos\theta-k')^2}{\text{A}^2z^2+\text{AM}(\beta^2-\alpha^2)^2\sin^2\theta\cos^2\theta}}\end{array}\right\}\ .\ (11)$$

and the velocities of its centre of gravity are

$$\frac{dx}{dt}=l', \frac{dy}{dt}=l', \frac{dz}{dt}$$

$$=(\beta^2-\alpha^2)\cos\theta\sqrt{\frac{\text{A}\sin^2\theta(k-2g\text{M}z)-(\text{c}k''\cos\theta-k')^2}{\text{A}^2z^2+\text{A M}(\beta^2-\alpha^2)^2\sin^2\theta\cos^2\theta}} \quad . \quad . \quad (12).$$

Hence all the velocities are given in terms of θ, and the time t may be found, by approximation, from the third of (11); and the loci of the point of contact and the centre of gravity from (8), and (7).

THIRD SOLUTION. *By Mr. Geo. R. Perkins, Clinton.*

In this solution I shall follow Mr. Poisson's notation, as given in the second volume of his Mechanics. Let the ellipsoid's equation be

$$\frac{\alpha^2}{a^2}+\frac{\beta^2}{a^2}+\frac{\gamma^2}{b^2}-1=\text{L}=0 \quad . \quad . \quad . \quad . \quad (1),$$

referred to its principal axes. Call K the point of contact with the plane, M its mass, G its centre of gravity; if to the weight of the ellipsoid we add the force R of unknown magnitude, which is the resistance of the plane against the ellipsoid, we may consider its centre of gravity as a free material point; if its co-ordinates are x_1, y_1, z_1, referred to fixed axes, of which the plane of x, y coincides with the given plane, and the axis of z is counted upwards, the differential equations of its motion will be

$$\text{M}\frac{d^2x_1}{dt^2}=0, \text{M}\frac{d^2y_1}{dt^2}=0, \text{M}\frac{d^2z_1}{dt^2}=\text{R}-\text{M}g \quad . \quad . \quad . \quad (2).$$

At the same time the ellipsoid will turn about G as about a fixed point, in virtue of the forces R and Mg applied at the points K and G, but the force Mg can have no influence on this rotation, since it passes through the point G, considered as fixed. Now the momenta of the force R, referred to the principal axes will be

$$\text{R}(\alpha b''-\beta a''), \text{R}(\gamma a''-\alpha c''), \text{R}(\beta c''-\gamma b'') \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3),$$

and the three equations of rotation are

$$\left.\begin{aligned} &\text{C}\,dr+(\text{B}-\text{A})pq\,dt=\text{R}(\alpha b''-\beta a'')dt,\\ &\text{B}\,dq+(\text{A}-\text{C})pr\,dt=\text{R}(\gamma a''-\alpha c'')dt,\\ &\text{A}\,dp+(\text{C}-\text{B})qr\,dt=\text{R}(\beta c''-\gamma b'')dt, \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4).$$

(——The symbols A, B, C; p, q, r; a'', b'', c'' have here precisely the same signification and relations with the angles θ, χ, φ, as in Dr. Strong's article on Rotation, and Mr. Perkins deduces from these equations, in the same manner as in the first solution—)

$$\text{A}\,a''p+\text{B}\,b''q+\text{C}c''r=l \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5),$$

$$\text{A}\,pdp+\text{B}\,qdq+\text{C}\,rdr=\text{R}(\alpha da''+\beta db''+\gamma dc'') \quad . \quad . \quad (6).$$

The equation of the given plane, referred to the principal axes, is

$$z_{,}+a''\alpha+b''\beta+c''\gamma=0,$$

hence $\alpha da''+\beta db''+\gamma dc''=-dz_{,}-(a''d\alpha+b''d\beta+c''d\gamma)$;

and, since the normal at the point K, is parallel to the fixed axis of $z_{,}$,

$$a''=\text{V}.\frac{d\text{L}}{d\alpha}, b''=\text{V}.\frac{d\text{L}}{d\beta}, c''=\text{V}.\frac{d\text{L}}{d\gamma}; \text{ where } \text{V}=\left\{\left(\frac{d\text{L}}{d\alpha}\right)^2+\left(\frac{d\text{L}}{d\beta}\right)^2+\left(\frac{d\text{L}}{d\gamma}\right)^2\right\}^{-\frac{1}{2}}$$

$$\therefore\quad a''d\alpha+b''d\beta+c''d\gamma=\text{V}.\left\{\frac{d\text{L}}{d\alpha}d\alpha+\frac{d\text{L}}{d\beta}d\beta+\frac{d\text{L}}{d\gamma}d\gamma\right\}=\text{V}d\text{L}=0,$$

and equation (6) becomes, by substituting this, and the value of n, and integrating

$$\text{A}p^2+\text{B}q^2+\text{C}r^2+\text{M}\left(\frac{dz_1^2}{dt^2}+2gz'\right)=h \quad . \quad . \quad . \quad . \quad (7);$$

also since the solid is one of revolution A=B, and $\alpha b''-\beta a''=0$, hence the first of equations (4) gives,

$$dr=0, \text{ and } r=\frac{d\varphi}{dt}-\cos\theta\frac{d\chi}{dt}=\text{const.}=n \quad . \quad . \quad . \quad . \quad . \quad . \quad (8).$$

By substitution, we shall also find

$$z_,=-\sqrt{a^2\sin^2\theta+b^2\cos^2\theta},\ \frac{dz_,}{dt}=\frac{(a^2-b^2)\sin\theta\cos\theta}{z_1}\cdot\frac{d\theta}{dt},$$

$$a''p+b''q=-\sin^2\theta\frac{d\chi}{dt},\ p^2+q^2=\sin^2\theta\frac{d\chi^2}{dt^2}+\frac{d\theta^2}{dt^2};$$

and equations (5) and (6) thus become

$$\text{C}n\cos\theta-\text{A}\sin^2\theta\frac{d\chi}{dt}=l, \quad . \quad . \quad . \quad . \quad . \quad (9)$$

$$\text{A}\left(\sin^2\theta\frac{d\chi^2}{dt^2}+\frac{d\theta^2}{dt^2}\right)+\text{M}\left\{\frac{(a^2-b^2)^2\sin^2\theta\cos^2\theta}{z_,^2}\cdot\frac{d\theta^2}{dt^2}+2gz_,\right\}=h \, . \, . \, (10),$$

where the constant h includes $-\text{C}n^2$. The equations (8), (9), (10) give the angular velocities of the solid. The equations of translation show that the projection of the path of the point G on the plane is a straight line uniformly described, and its distance from the plane is $z_,=-\sqrt{a^2\sin^2\theta+b^2\cos^2\theta}$.

——Professor Peirce's solutions to the last three questions did not arrive until this part of the copy was prepared for the press. His results confirm those of the gentlemen whose solutions are published.——

List of Contributors, and of Questions answered by each. The figures refer to the number of the Questions, as marked in Number V. Art. XX.

PROFESSOR C. AVERY, Hamilton College, N. Y. ans. all the questions,
PROFESSOR F. N. BENEDICT, University of Vermont, ans. 9.
P. BARTON, JR., Athol, Massachusetts, ans. 4.
B. BIRDSALL, New-Hartford, N. Y., ans. 1, 2, 3, 4, 11.
PROFESSOR M. CATLIN, Hamilton College, N. Y., ans. all the questions,
R. S HOWLAND, Junior Class, St. Paul's College, ans. 1, 3.
INVESTIGATOR, ans. 3.
WILLIAM LENHART, York, Penn., ans. 10,
J. F. MACULLY, Teacher of Mathematics, New-York, ans. 1, 2, 11.
PROFESSOR B. PEIRCE, Harvard University, Mass., ans. all the questions.
GEORGE R. PERKINS, Clinton Liberal Institute, ans. all the questions,
ψ, ans. 5.
O. ROOT, Principal of Syracuse Acad., N. Y., ans. 1, 2, 3, 4, 5, 9, 11, 12, 14.
PROF. T. STRONG, LL. D., New Brunswick, N. J., ans. all the questions.

*** All communications for Number VIII. which will be published on the first day of November, 1839, must be post paid, addressed to the Editor, St. Paul's College, Flushing, L. I, and must arrive before the first of August, 1839. New Questions must be accompanied with their solutions.

☞ Mr. George R. Perkins, of Clinton, Oneida Co., N. Y. is desirous of obtaining a situation as Teacher of Mathematics. For further particulars reference may be had to the following gentlemen: Hon. John A. Dix, Albany; Hon. John H. Prentiss, Cooperstown, Otsego Co., N. Y.; Rev. George B. Miller, D. D., Principal of Hartswick Seminary, Otsego Co., N. Y.; Rev. C. B. Thummel, A. M., Georgetown, S. C.

*** We have been obliged, for want of room in this number, to defer the insertion of Mr. Lenhart's Diophantine speculations, No. III.; as well as an article on the application of Sturm's Theorem to the general equations of the 4th and 5th degrees, by Δ.

ARTICLE II.

NEW QUESTIONS TO BE ANSWERED IN NUMBER IX.

Their solutions must arrive before February 1st, 1840.

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(113). QUESTION I. *By Prof. N. May, M. D., St. Paul's College.*

What degree of temperature will be indicated by the same number on two scales, attached to the same thermometer, and graduated, the one by the centigrade, the other by Fahrenheit's, method of division.

[illegible] p. 10.

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(114). QUESTION II. *From Peirce's Algebra.*

Solve the two equations

$$x^2y^4 - 8y^2x^2 + 16x^2 = 90xy + 60(x - y^2) - 720(y - 1),$$
$$\frac{(y^2 - 4y + 4)x}{5} = 3 - \frac{12}{x}.$$

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(115). QUESTION III. *By P.*

Find the value of the infinitely continued fraction

$$\sqrt[n]{a + \cfrac{b}{\sqrt[n]{a + \cfrac{b}{\sqrt[n]{a + \cfrac{b}{\sqrt[n]{a + \&c.,}}}}}}}$$

[illegible] p. 11.

and give an example when $a = 23$, $b = 10$, $n = 2$.

(116). QUESTION IV. *By Mr. Sam. J. Gummere, Haverford School, Pa.*

Find the value of $x^{\frac{1}{x}}$, when $x = 0$.

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(117). QUESTION V. *By J. F. Macully, Esq., New-York.*

Points are taken on the plane of a given triangle, so that the sum of the squares of the three perpendiculars drawn from any one point to the sides, shall be equal to the area of the triangle. These points will all be found in the periphery of an ellipse, whose position and magnitude are required.

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(118). QUESTION VI. *By Mr. P. Barton, jun., Athol, Mass.*

It is required to circumscribe the least isosceles triangle about two circles, touching each other, and the base of the triangle; the diameters of the circles being 16 and 20.

(119.) QUESTION VII. *By Wm. Lenhart, Esq., York, Pa.*

Find parallelograms whose sides and diagonals are integers.

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(120). QUESTION VIII. *By J. F. Macully, Esq., New-York.*

Find the sum of n terms of the series

$$\frac{1+2\sin^2\frac{1}{2}\theta}{(1-2\cos\theta)^2}+\frac{1}{9}\cdot\frac{1+2\sin^2\frac{1}{6}\theta}{(1-2\cos\frac{1}{3}\theta)^2}+\frac{1}{9^2}\cdot\frac{1+2\sin^2\frac{1}{18}\theta}{(1-2\cos\frac{1}{9}\theta)^2}+\text{\&c.}$$

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(121). QUESTION IX. *By Mr. J. S. Van de Graaff, Lexington, Ken.*
(From the Mathematical Diary.)

To find, on the surface of a given sphere, the area of the greatest triangle, whose perimeter is a semi-circumference, and whose greater angle is just double the smaller.

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(122). QUESTION X. *By Investigator.*

Two bodies of given masses, and composed of matter the particles of which attract each other with forces inversely proportional to the squares of their distances, are placed at a given distance from each other, and then projected in *opposite* directions along the same straight line. It is required to find all the circumstances of their motion until they are in contact.

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(123). QUESTION XI. *By Mr. W. S. B. Woolhouse, London.*

Sum the series

$$\frac{xy}{x+y}+\frac{x^2y^2}{(x+y)(x^2+y^2)}+\frac{x^4y^4}{(x+y)(x^2+y^2)(x^4+y^4)}+\text{\&c.}$$

(124). QUESTION XII. *By Mr. George R. Perkins, Clinton, N. Y.*

Ansr. Misc. p. 106.

a_1	a_2	a_3	a_4
b_1	b_2	b_3	b_4
c_1	c_2	c_3	c_4
d_1	d_2	d_3	d_4

The points A, B, C, D and A′, B′, C′, D′ correspond severally with the vertices of any two regular tetraedrons inscribed within a given sphere, and the cosines of the arcs drawn from A′ to the points A, B, C, D are represented by a_1, a_2, a_3, a_4; the cosines of the arcs drawn from B′, to the same points, by b_1, b_2, b_3, b_4; and similarly for those drawn from C′ and D′ to the same points, these cosines being arranged as in the annexed square. It is required to show that the sum of the squares of the four cosines in any horizontal line, as well as in any vertical line, is equal to $\frac{4}{3}$; and that the sum of the products two and two, such as $a_1 b_1 + a_2 b_2 + a_3 b_3 + a_4 b_4$ with regard to any two horizontal lines, as well as with regard to any two vertical lines, is equal to $-\frac{4}{9}$.

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(125). QUESTION XIII. *By* Δ.

If from a given point in the transverse axis of a spherical ellipse, (Vol. I, page 184,) perpendicular arcs be let fall upon the great circle tangents of the curve; the vertices of the right angles will be in a spherical curve, whose equation and properties are required.

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(126). QUESTION XIV. *By* Δ.

It is required to find the equations of the surfaces described in questions (17) and (18), Vol. I. pp. 95 and 97, or to integrate the partial differential equations

Coll. ii. p. 18

$$1^\circ.\ (z - px - qy)^2(1 - p - q) = A^2 pq,$$
$$2^\circ.\ (z - px - qy)^4(1 + p^2 + q^2) = A^4(t^2 - p^2 - q^2)^3;$$

in which, t and A are constants, $p = \frac{dz}{dx}$, $q = \frac{dz}{dy}$.

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(127). QUESTION XV. *By Prof. B. Peirce, Harvard University, Mass.*

Supposing a very light body, of such a substance as not to be impeded in its motions by collision with the earth, to be moving in an orbit in the plane of the ecliptic, with a time of revolution nearly half that of the earth, and an aphelion distance nearly equal to the distance of the earth from the sun, which we will suppose to be constant, and that at the instant of its passing the aphelion it is nearly in conjunction with the earth; to find the perturbations in its motion as caused by the earth.

ARTICLE III.

MOTION OF A SYSTEM OF BODIES ~~ABOUT~~ round A FIXED ~~AXIS~~ point.

By Prof. T. Strong, LL. D., New-Brunswick, N. J.

1. We will commence this paper by showing how to change the rectangular co-ordinates of a point when referred to one system of axes, to those referred to another system, having the same origin.

Let x, y, z denote the co-ordinates of the point referred to in the first system, and x', y', z', those of the same point referred to in the second system, the origin being the same; let L denote the right line drawn from the origin to the point; a, b, c the cosines of the angles which the axis of x makes with the axes of x', y', z'; a' b' c' the corresponding cosines for the axis of y, and a'', b'', c'' those for the axis of z.

It is evident that x=the projection of L on the axis of x,
=the sum of the projections of x', y', z' on that axis,
and similarly for y and z; hence

$$x = ax' + by' + cz',\ y = a'x' + b'y' + c'z',\ z = a''x' + b''y' + c''z' \quad . \ (1).$$

Since $\qquad L^2 = x^2 + y^2 + z^2 = x'^2 + y'^2 + z'^2$;

if we substitute the values of x, y, z from (1), the resulting equation will be an identical one, true for all values of x', y', z', and therefore, by the principle of indeterminate co-efficients,

$$\left.\begin{array}{lll} a^2 + a'^2 + a''^2 = 1, & b^2 + b'^2 + b''^2 = 1, & c^2 + c'^2 + c''^2 = 1; \\ ab + a'b' + a''b'' = 0, & ac + a'c' + a''c'' = 0, & bc + b'c' + b''c'' = 0. \end{array}\right\} \ (2).$$

If we now multiply the equations in (1) by a, a', a'' severally, and add the results; then successively by b, b', b'' and c, c', c'', we get

$$x' = ax + a'y + a''z,\ y' = bx + b'y + b''z,\ z' = cx + c'y + c''z \quad . \ (3);$$

from which we get, as before,

$$\left.\begin{array}{lll} a^2 + b^2 + c^2 = 1, & a'^2 + b'^2 + c'^2 = 1, & a''^2 + b''^2 + c'' = 1; \\ aa' + bb' + cc' = 0, & aa'' + bb'' + cc'' = 0, & a'a'' + b'b'' + c'c'' = 0. \end{array}\right\} . \ (4),$$

which are evidently equivalent to (2), and therefore only three of the nine cosines, a, b, c, a' &c. are arbitrary. Since, by (2)

$$bc + b'c' + b''c'' = 0,\ c'^2 + c''^2 = 1 - c^2,\ b'^2 + b''^2 = 1 - b^2,$$

we get

$$\begin{aligned} b^2 + c^2 &= b^2 + c^2 - 2bc(bc + b'c' + b''c'') \\ &= b^2(1 - c^2) + c^2(1 - b^2) - 2bc(b'c' + b''c'') \\ &= b^2(c'^2 + c''^2) + c^2(b'^2 + b''^2) - 2bc(b'c' + b''c'') \\ &= (bc' - cb')^2 + (bc'' - b''c)^2 \\ &= 1 - (b'c'' - c'b'')^2, \end{aligned}$$

since we have, identically,

$$(bc' - cb')^2 + (bc'' - b''c)^2 + (b'c'' - b''c')^2$$
$$= (b^2 + b'^2 + b''^2)(c^2 + c'^2 + c''^2) - (bc + b'c' + b''c'')^2 = 1.$$

But, by (4), $\qquad b^2 + c^2 = 1 - a^2,$

therefore, $\ a^2 = (b'c'' - c'b'')^2$, and $a = \pm(b'c'' - c'b'')$;

the sign (—) does not, however, apply, for supposing the co-ordinates x, y, z to coincide with x', y', z', we have $a=1,\ b'=1,\ c''=1,\ b''=0,\ c''=0$ therefore

$$a = b'c'' - b''c',$$

and in a similar manner we get the equations

$$\left.\begin{array}{l} a = b'c'' - b''c',\ b = a''c' - a'c'',\ c = a'b'' - a''b', \\ a' = b''c - bc'',\ b' = ac'' - a''c,\ c' = a''b - ab'', \\ a'' = bc' - b'c,\ b'' = ac' - a'c,\ c'' = ab' - a'b, \end{array}\right\} \quad . \quad (5).$$

2. We will now show how to find values of a, b, c, a', &c., which will satisfy the equations of condition thus obtained.

Imagine (with La Place, Mec. Cel., p. 58, or Com. p. 111,) that the origin of the co-ordinates is at the centre of the earth, that x, y are in the plane of the ecliptic, x', y' in that of the equator, and that the axes of z, z' are drawn to the north poles of the ecliptic and equator; let χ, $\frac{1}{2}\pi + \chi$ denote the angles made by the axes of x and y, with the earth's radius drawn to the vernal equinox, these angles being reckoned according to the order of the signs of the zodiac; let φ, $\frac{1}{2}\pi + \varphi$ denote the angles which the axes of x' and y' make with the same radius, counted in direction of the earth's rotation about its axis; and let θ denote the obliquity of the ecliptic = the angle made by the axes of z and z'. It is evident that the sum of the projections of x, y, z on any right line = the sum of the projections of x', y', z' on that line, since they are each = the projection of L on that line; hence, if we project the two systems of co-ordinates on the line of the equinoxes, the projections of z and z' are each = 0, and we have

$$x \cos \chi - y \cos(\tfrac{1}{2}\pi + \chi) = x' \cos \varphi + y' \cos(\tfrac{1}{2}\pi + \varphi),$$

or

$$x \cos \chi - y \sin \chi = x' \cos \varphi - y' \sin \varphi \quad . \quad . \quad . \quad . \quad . \quad (6).$$

Similarly, if the two systems be projected on the line of the solstices,

$$x \sin \chi + y \cos \chi = (x' \sin \varphi + y' \cos \varphi) \cos \theta + z' \sin \theta \quad . \quad . \quad (7);$$

and if the systems be projected on the line of intersection of the plane of z, z', or solstitial colure, and that of the equator,

$$(x \sin \chi + y \cos \chi) \cos \theta - z \sin \theta = x' \sin \varphi + y' \cos \varphi \quad . \quad . \quad (8).$$

By a very obvious reduction, we get from these equations

$$\begin{aligned} x &= x'(\cos\theta \sin\chi \sin\varphi + \cos\chi \cos\varphi) + y'(\cos\theta \sin\chi \cos\varphi - \cos\chi \sin\varphi) + z' \sin\theta \sin\chi, \\ y &= x'(\cos\theta \cos\chi \sin\varphi - \sin\chi \cos\varphi) + y'(\cos\theta \cos\chi \cos\varphi + \sin\chi \sin\varphi) + z' \sin\theta \cos\chi, \\ z &= -x' \sin\theta \sin\varphi - y' \sin\theta \cos\varphi + z' \cos\theta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9), \end{aligned}$$

which agree with those of Laplace, at the place cited.

By comparing the values of x, y, z in (1) and (9), we get

$$\begin{aligned} a &= \cos\theta \sin\chi \sin\varphi + \cos\chi \cos\varphi,\ b = \cos\theta \sin\chi \cos\varphi - \cos\chi \sin\varphi,\ c = \sin\theta \sin\chi; \\ a' &= \cos\theta \cos\chi \sin\varphi - \sin\chi \cos\varphi,\ b' = \cos\theta \cos\chi \cos\varphi + \sin\chi \sin\varphi,\ c' = \sin\theta \cos\chi; \\ a'' &= -\sin\theta \sin\varphi, \qquad b'' = -\sin\theta \cos\varphi, \qquad c'' = \cos\theta \ (10), \end{aligned}$$

which, on trial, will be found to fulfil the equations of condition (2), (4), (5); so that χ, φ, θ will be indeterminates, as they ought to be.

We shall now suppose that a system of bodies m, m' m'', &c. is revolving about any fixed point, and shall indefinitely denote any body of the system, by m; we shall denote by x, y, z the co-ordinates of m, when referred to any system of rectangular axes, fixed in space, at the time t from any given epoch; let also P, Q, R be the resultants of all the forces that affect an unit of m, when decomposed in the direction of x, y, z severally, these forces being supposed to tend to increase the co-ordinates; then we shall have

$$m \frac{d^2 x}{dt^2} = m\text{P},\ m \frac{d^2 y}{dt^2} = m\text{Q},\ m \frac{d^2 z}{dt^2} = m\text{R},$$

and using the sign of finite integrals, S, to denote the sum of all the equations thus formed for each body in the system,

$$Sm\frac{d^2x}{dt^2} = Sm\text{P},\ Sm\frac{d^2y}{dt^2} = Sm\text{Q},\ Sm\frac{d^2z}{dt^2} = Sm\text{R} \quad . \quad . \ (a),$$

where we suppose that m, m', &c. are each so small, that the co-ordinates of all their points may be regarded as the same: we shall also have

$$\left.\begin{aligned} d\,.\,Sm\left(\frac{xdy - ydx}{dt}\right) &= dt\,.\,Sm(\text{Q}x - \text{P}y) = d\text{N}''', \\ d\,.\,Sm\left(\frac{zdx - xdz}{dt}\right) &= dt\,.\,Sm(\text{P}z - \text{R}x) = d\text{N}'', \\ d\,.\,Sm\left(\frac{ydz - zdy}{dt}\right) &= dt\,.\,Sm(\text{R}y - \text{Q}z) = d\text{N}', \end{aligned}\right\} \quad . \quad (b)$$

which are the formulæ of rotation that we propose to transform into others which shall be more convenient in practice.

We shall now suppose that the co-ordinates of m, when referred to any other system, having the same origin, are x', y', z'; then if we suppose $x', y' z'$ as well as a, b, c, a', &c. are functions of t; we shall have from (1),

$$\left.\begin{aligned} \frac{dx}{dt} &= \frac{x'da + y'db + z'dc}{dt} + \frac{adx' + bdy' + cdz'}{dt}, \\ \frac{dy}{dt} &= \frac{x'da' + y'db' + z'dc'}{dt} + \frac{a'dx' + b'dy' + c'dz'}{dt} \\ \frac{dz}{dt} &= \frac{x'da'' + y'db'' + z'dc''}{dt} + \frac{a''dx' + b''dy' + c''dz'}{dt} \end{aligned}\right\} \quad . \quad (c).$$

Put

$cdb + c'db' + c''db'' = pdt,\ adc + a'dc' + a''dc'' = qdt,\ bda + b'da' + b''da'' = rdt;$

then, by (2), we shall also have

$$\left.\begin{aligned} bdc + b'dc' + b''dc'' &= -pdt, \\ cda + c'da' + c''da'' &= -qdt, \\ adb + a'db' + a''db'' &= -rdt, \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \ (d);$$

and by substituting the values of a, b, c, a', &c., from (10) in (d), we have

$$\left.\begin{aligned} \sin\varphi \sin\theta d\chi - \cos\varphi d\theta &= pdt, \\ \cos\varphi \sin\theta d\chi + \sin\varphi d\theta &= qdt, \\ d\varphi - \cos\theta d\chi &= rdt, \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \ (e).$$

Put, $qz' - ry' + \dfrac{dx'}{dt} = \text{L},\ rx' - pz' + \dfrac{dy'}{dt} = \text{M},\ py' - qx' + \dfrac{dz'}{dt} = \text{N} \ . \ (f)$;

multiply the equations in (c) by a, a', a'',; then by b, b', b''; then by c, c', c''; adding the respective results, and reducing by (d), (2), (4); we get

$$\frac{adx + a'dy + a''dz}{dt} = \text{L},\ \frac{bdx + b'dy + b''dz}{dt} = \text{M},\ \frac{cdx + c'dy + c''dz}{dt} = \text{N}\ (g).$$

Multiply these severally by a, b, c; then by a', b', c'; then by a'', b'', c'', adding the products, and we get

$$\frac{dx}{dt} = a\text{L} + b\text{M} + c\text{N},\ \frac{dy}{dt} = a'\text{L} + b'\text{M} + c'\text{N},\ \frac{dz}{dt} = a''\text{L} + b''\text{M} + c''\text{N} \ . \ (h).$$

The sum of the squares of these equations is

$$\frac{dx^2+dy^2+dz^2}{dt^2}=\text{L}^2+\text{M}^2+\text{N}^2$$

$$=\frac{dx'^2+dy'^2+dz'^2}{dt^2}+(x'^2+y'^2)r^2+(x'^2+z'^2)q^2+(y'^2+z'^2)p^2$$

$$-2x'y'pq-2x'z'pr-2y'z'qr$$

$$+2p\left(\frac{y'dz'-z'dy'}{dt}\right)+2q\left(\frac{z'dx'-x'dz'}{dt}\right)+2r\left(\frac{x'dy'-y'dx'}{dt}\right) \quad (i).$$

Put

$$\left.\begin{array}{l} S(y'^2+z'^2)m=\text{A},\ Sy'z'm=\text{D},\ Sm\left(\frac{x'dy'-y'dx'}{dt}\right)=\text{A}'; \\ S(x'^2+z'^2)m=\text{B},\ Sx'z'm=\text{E},\ Sm\left(\frac{z'dx'-x'dz'}{dt}\right)=\text{B}'; \\ S(x'^2+y'^2)m=\text{C},\ Sx'y'm=\text{F},\ Sm\left(\frac{y'dz'-z'dy'}{dt}\right)=\text{C}'; \\ \text{A}p-\text{E}r-\text{F}q=p',\ \text{B}q-\text{D}r-\text{F}p=q',\ \text{C}r-\text{D}q-\text{E}p=r' \end{array}\right\} (k),$$

then we easily deduce from (f), after reducing by (5) and by (k),

$$Sm(\text{N}y'-\text{M}z')=p'+\text{C}',\ Sm(\text{L}z'-\text{N}x')=q'+\text{B}',\ Sm(\text{M}x'-\text{L}y')=r'+\text{A}',$$

and if we substitute these, together with the values of x, y, z from (1), and those of $\frac{dx}{dt}$, $\frac{dy}{dt}$, $\frac{dz}{dt}$ from (h), in the three equations (b), we get

$$\left.\begin{array}{l} d.\left[a''(p'+\text{C}')+b''(q'+\text{B}')+c''(r'+\text{A}')\right]=d\text{N}''', \\ d.\left[a'(p'+\text{C}')+b'(q'+\text{B}')+c'(r'+\text{A}')\right]=d\text{N}'', \\ d.\left[a(p'+\text{C}')+b(q'+\text{B}')+c(r'+\text{A}')\right]=d\text{N}', \end{array}\right\} \quad . \quad (l);$$

hence

$$\left.\begin{array}{l} \frac{d(p'+\text{C}')}{dt}+q(r'+\text{A}')-r(q'+\text{B}')=\frac{a''d\text{N}'''+a'd\text{N}''+ad\text{N}'}{dt}, \\ \frac{d(q'+\text{B}')}{dt}+r(p'+\text{C}')-p(r'+\text{A}')=\frac{b''d\text{N}'''+b'd\text{N}''+bd\text{N}'}{dt}, \\ \frac{d(r'+\text{A}')}{dt}+p(q'+\text{B}')-q(p'+\text{C}')=\frac{c''d\text{N}'''+c'd\text{N}''+cd\text{N}'}{dt}, \end{array}\right\} (m).$$

Put the second members of equations (m) = $\text{N}_{,}$, $\text{N}_{,,}$, $\text{N}_{,,,}$ respectively, then by (b),

$$\text{N}_{,}=Sm[(a'z-a''y)\text{P}+(a''x-az)\text{Q}+(ay-a'x)\text{R}],$$

and similarly the others. But, by (1) and (5),

$$a'z-a''y=cy'-bz',\ a''x-az=c'y'-b'z',\ ay-a'x=c''y'-b''z';$$

therefore put

$$c\text{P}+c'\text{Q}+c''\text{R}=\text{R}',\ b\text{P}+b'\text{Q}+b''\text{R}=\text{Q}',\ a\text{P}+a'\text{Q}+a''\text{R}=\text{P}' \quad . \quad . \quad (n),$$

then $\text{N}_{,}=Sm(\text{R}'y'-\text{Q}'z')$, $\text{N}_{,,}=Sm(\text{P}'z'-\text{R}'x')$, $\text{N}_{,,,}=Sm(\text{Q}'x'-\text{P}'y')$. (o),

Multiply (i) by (m), and take the finite integrals relative to all the bodies of the system, putting $Sm\left(\frac{dx^2+dy^2+dz^2}{dt^2}\right)=2\text{T}$ = the living force of the system; then

$$T = \tfrac{1}{2}(Ap^2 + Bq^2 + Cr^2) - Dqr - Epr - Fpq + C'p + B'q + A'r + Sm\left(\frac{dx'^2 + dy'^2 + dz'^2}{2dt^2}\right) \quad . \quad . \quad (p);$$

By taking the partial differential co-efficients of (p), relative to p, q, r, and noticing (k), we get

$$\frac{dT}{dp} = p' + C', \frac{dT}{dq} = q' + B', \frac{dT}{dr} = r' + A' \quad . \quad . \quad . \quad . \quad (q);$$

hence (m) will be changed to

$$\left.\begin{aligned} &\frac{d\left(\frac{dT}{dp}\right)}{dt} + q\left(\frac{dT}{dr}\right) - r\left(\frac{dT}{dq}\right) = Sm(R'y' - Q'z'), \\ &\frac{d\left(\frac{dT}{dq}\right)}{dt} + r\left(\frac{dT}{dp}\right) - p\left(\frac{dT}{dr}\right) = Sm(P'z' - R'x'), \\ &\frac{d\left(\frac{dT}{dr}\right)}{dt} + p\left(\frac{dT}{dq}\right) - q\left(\frac{dT}{dp}\right) = Sm(Q'x' - P'y'), \end{aligned}\right\} \quad . \quad . \quad (r),$$

which agree with the equations given by La Grange, Mec. Analytique, vol. 2, p. 365, edition of 1815. Also, if we substitute (q) in (l), they become

$$\left.\begin{aligned} &d.\left[a''\left(\frac{dT}{dp}\right) + b''\left(\frac{dT}{dq}\right) + c''\left(\frac{dT}{dr}\right)\right] = dN''', \\ &d.\left[a'\left(\frac{dT}{dp}\right) + b'\left(\frac{dT}{dq}\right) + c'\left(\frac{dT}{dr}\right)\right] = dN'', \\ &d.\left[a\left(\frac{dT}{dp}\right) + b\left(\frac{dT}{dq}\right) + c\left(\frac{dT}{dr}\right)\right] = dN', \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (s).$$

Now it may be proved, as in p. 55, &c. Vol. I., Mec. Cel., or Com. p. 104, &c. that (b) are independent of the mutual action of the bodies of the system on each other, and of any forces that are directed to or from the origin of co-ordinates; therefore if the system is not acted on by any foreign forces, except those passing through the origin of co-ordinates, and by the reciprocal action of the bodies that compose it; we shall have, by (b), and by integration,

$$d.Sm\left(\frac{xdy - ydx}{dt}\right) = 0, \ Sm\left(\frac{xdy - ydx}{dt}\right) = A'';$$

$$d.Sm\left(\frac{xdz - zdx}{dt}\right) = 0, \ Sm\left(\frac{xdz - zdx}{dt}\right) = B'';$$

$$d.Sm\left(\frac{ydz - zdy}{dt}\right) = 0, \ Sm\left(\frac{ydz - zdy}{dt}\right) = C'';$$

A'', B'', C'' being the arbitrary constants; but it is evident, by the method of obtaining (l), that we have

$$\frac{xdy - ydx}{dt} = a''(p' + C') + b''(q' + B') + c''(r' + A'),$$

and similarly for the like quantities; hence, if a, b, c, a' &c. are supposed invariable, and consequently $p'=0$, $q'=0$, $r'=0$, we have

$$a''\text{C}'+b''\text{B}'+c''\text{A}'=\text{A}'',\ a'\text{C}'+b'\text{B}'+c'\text{A}'=\text{B}'',\ ac'+b'\text{B}'+c'\text{A}'=\text{A}'' \quad . \quad (t).$$

But if we observe the values of A', B', C' in (K), and remark that, on the suppositions here made, the co-ordinates x', y', z' are fixed in position, and have the same origin as the other system of co-ordinates; we shall see that A', B', C' are constant, as we have before shown that A'', B'', C'' are so; and if we add the squares of (t)

$$\text{A}'^2+\text{B}'^2+\text{C}'^2=\text{A}''^2+\text{B}''^2+\text{C}''^2 \quad . \quad . \quad . \quad . \quad . \quad (u);$$

we also get from (t)

$$\text{A}'=c''\text{A}''+c'\text{B}''+c\text{C}'',\ \text{B}'=b''\text{A}''+b'\text{B}''+b\text{C}'',\ \text{C}'=a''\text{A}''+a'\text{B}''+a\text{C}'' \quad . \quad (v).$$

Now, the position of the plane $x'y'$ is arbitrary, let it be assumed so that

$$c''=\frac{\text{A}''}{\text{V}},\ c'=\frac{\text{B}''}{\text{V}},\ c=\frac{\text{C}''}{\text{V}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (w),$$

where $\qquad \text{V}=\sqrt{\text{A}''^2+\text{B}''^2+\text{C}''^2}$;

then, by (u), (v) and (w), we find

$$\text{A}'^2=\text{A}''^2+\text{B}''^2+\text{C}''^2,\ \text{B}'=0,\ \text{C}'=0;$$

and substituting (w) in (10), we get

$$\cos\theta=\frac{\text{A}''}{\text{V}},\ \sin\theta\cos\chi=\frac{\text{B}''}{\text{V}},\ \sin\theta\sin\chi=\frac{\text{C}''}{\text{V}} \quad . \quad . \quad . \quad . \quad (x).$$

The equations (w) agree with the formulæ given at p. 269, Vol. I., Mec. Anal., for the determination of the invariable plane, and (x) are given for the same purpose at p. 60, Vol. I., Mec. Cel., Com. p. 120.

Again, suppose the system to be rigid, and that the axes of x', y', z' are firmly connected with it, so that they do not vary with the time, and change their values only in passing from one body to another, then $\frac{dx'}{dt}$, $\frac{dy'}{dt}$, $\frac{dz'}{dt}$ each $=0$, therefore $\text{A}'=\text{B}'=\text{C}'=0$; hence ($m$) become

$$\frac{dp'}{dt}+qr'-rq'=\text{N}_{,},\ \frac{dq'}{dt}+rp'-pr'=\text{N}_{,,},\ \frac{dr'}{dt}+pq'-qp'=\text{N}_{,,,} \,.\, (y);$$

if the axes of x', y', z' are principal axes, $\text{D}=\text{E}=\text{F}=0$, and by restoring the values of p', q', r', equations (y) become

$$\text{A}\,.\frac{dp}{dt}+(\text{C}-\text{B})qr=\text{N}_{,},\ \text{B}\,.\frac{dq}{dt}+(\text{A}-\text{C})pr=\text{N}_{,,},\ \text{C}.\frac{dr}{dt}+(\text{B}-\text{A})pq=\text{N}_{,,,} \quad . \ (z),$$

or, writing the values of $\text{N}_{,}$, $\text{N}_{,,}$, $\text{N}_{,,,}$ as in (o), they become

$$\left.\begin{aligned}\text{A}\frac{dp}{dt}+(\text{C}-\text{B})qr&=\text{S}m(\text{R}'y'-\text{Q}'z'),\\ \text{B}\frac{dq}{dt}+(\text{A}-\text{C})pr&=\text{S}m(\text{P}'z'-\text{R}'x'),\\ \text{C}\frac{dr}{dt}+(\text{B}-\text{A})pq&=\text{S}m(\text{Q}'x'-\text{P}'y'),\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (z').$$

Since the position of the axes of x, in the plane xy, is arbitrary, we will suppose that it makes an indefinitely small angle with the line of intersection of the planes xy and $x'y'$; hence neglecting infinitely small

quantities of the second, &c., orders, $\cos \chi = 1$, $\sin \chi = \chi$. Therefore, neglecting quantities of the order χ, in a'', a', &c., as given in (10), restoring the values of $N_{,}$, $N_{,,}$, $N_{,,,}$, and multiply the equations in (z) by dt, they become

$$\left.\begin{array}{l} \mathrm{A}dp + (\mathrm{C} - \mathrm{B})qrdt = -(\sin\theta dN''' - \cos\theta dN'')\sin\varphi + \cos\varphi dN', \\ \mathrm{B}dq + (\mathrm{A} - \mathrm{C})prdt = -(\sin\theta dN''' - \cos\theta dN'')\cos\varphi - \sin\varphi dN', \\ \mathrm{C}dr + (\mathrm{B} - \mathrm{A})pqdt = \quad \cos\theta dN''' + \sin\theta dN'' \end{array}\right\} (z''),$$

which agree with (D) given at p. 74, Vol. I., Mec. Cel., or Com. p. 157. It is evident that all the formulæ which are applicable in the case of a rigid system, become applicable to the motion of a continuous solid, by changing m into dm, and then integrating relative to its mass in the expressions for A, B, &c., which in the rigid system depend on the integrals of finite differences.

Finally, it is evident that the system may be considered as having a momentary axis of rotation, and that whether it is a rigid system or a continuous solid; to find the momentary axis, we observe that, relative to it, we have $\frac{dx}{dt} = 0$, $\frac{dy}{dt} = 0$, $\frac{dz}{dt} = 0$; also since the system is rigid, or a solid, $\frac{dx'}{dt} = \frac{dy'}{dt} = \frac{dz'}{dt} = 0$; therefore, by ($g$), $\mathrm{L} = \mathrm{M} = \mathrm{N} = 0$, or, by ($f$),

$$qz' - ry' = 0,\ rx' - pz' = 0,\ py' - qx' = 0;$$

which shows that the momentary axis passes through the origin of coordinates. If $a_{,}$, $b_{,}$, $c_{,}$ denote the cosines of the angles which the momentary axis makes with the axes of x', y', z', we shall have

$$\frac{p}{\sqrt{p^2+q^2+r^2}} = \frac{x'}{\sqrt{x'^2+y'^2+z'^2}} = a_{,},$$

$$\frac{q}{\sqrt{p^2+q^2+r^2}} = \frac{y'}{\sqrt{x'^2+y'^2+z'^2}} = b_{,},$$

$$\frac{r}{\sqrt{p^2+q^2+r^2}} = \frac{z'}{\sqrt{x'^2+y'^2+z'^2}} = c_{,};$$

which determine the position of this axis.

Since $\frac{dx'}{dt} = \frac{dy'}{dt} = \frac{dz'}{dt} = 0$, if we suppose $x' = y' = 0$, we get by (i),

$$z'\sqrt{p^2+q^2} = \frac{\sqrt{dx^2+dy^2+dz^2}}{dt},$$

for the velocity of a point on the axis of z', at the distance z' from the origin; but we have $\sqrt{\frac{p^2+q^2}{p^2+q^2+r^2}} = \sqrt{1-c_{,}^2}$ = sine of the angle made by the momentary axis with that of z'; therefore the perpendicular from the extremity of z' to the instantaneous axis is $= z'\sqrt{\frac{p^2+q^2}{p^2+q^2+r^2}}$; put ω = the angular velocity round the momentary axis, and we shall have

$$\omega = z'\sqrt{(p^2+q^2)} \div z'\sqrt{\frac{p^2+q^2}{p^2+q^2+r^2}} = \sqrt{p^2+q^2+r^2};$$

therefore $\qquad p = a_{,}\omega,\ q = b_{,}\omega,\ r = c_{,}\omega,$

where p, q, r are evidently the momentary rotations round the axes of x', y', z' severally; and therefore rotary velocities are compounded and resolved by the same rules as rectilineal ones.

Remarks. If in (a) we put $\mathrm{x}+x, \mathrm{y}+y, \mathrm{z}+z$ for x, y, z, and suppose $\mathrm{x}, \mathrm{y}, \mathrm{z}$ to denote the co-ordinates of the centre of gravity of the system, then, by the nature of that point, we have

$$Smx = 0,\ Smy = 0,\ Smz = 0;$$

therefore $$Sm\frac{d^2x}{dt^2} = 0,\ Sm\frac{d^2y}{dt^2} = 0,\ Sm\frac{d^2z}{dt^2} = 0 \quad . \quad . \quad . \quad (a'),$$

and (a) become $$\frac{d^2\mathrm{x}}{dt^2} = \frac{Sm\mathrm{P}}{Sm},\ \frac{d^2\mathrm{y}}{dt^2} = \frac{Sm\mathrm{Q}}{Sm},\ \frac{d^2\mathrm{z}}{dt^2} = \frac{Sm\mathrm{R}}{Sm} \quad . \quad . \quad . \quad (b')$$

and the centre of gravity moves as if all the bodies were united at that point, and the same forces were applied to them, and in the same manner.

Again if we substitute in (b), $\mathrm{x}+x, \mathrm{y}+y, \mathrm{z}+z$, for x, y, z; the terms depending on $\mathrm{x}, \mathrm{y}, \mathrm{z}$, will vanish from the equations, which will remain the same as before; and therefore placing the origin of co-ordinates, x, y, z in the centre of gravity of the system, the equations of rotation we have found will obtain, whether the centre of gravity is at rest or in motion.

Moreover, for a rigid system, in which x', y', z' are independent of t, if the values of $\frac{dx}{dt}$ given in (c) and (h) be compared, the values of $\mathrm{L}, \mathrm{M}, \mathrm{N}$ being substituted from (f), we shall have

$$x'\frac{da}{dt} + y'\frac{db}{dt} + z'\frac{dc}{dt} = (br - cq)x' + (cp - ar)y' + (aq - bp)z',$$

this, with the similar equations had by comparing the values of $\frac{dy}{dt}, \frac{dz}{dt}$, are identical equations, and the co-efficients of x', y', z' are equal; then

$$\left.\begin{aligned} da &= (b\,r - c\,q)dt, & db &= (c\,p - a\,r)dt, & dc &= (a\,q - b\,p)dt, \\ da' &= (b'r - c'q)dt, & db' &= (c'p - a'r)dt, & dc' &= (a'q - b'p)dt, \\ da'' &= (b''r - c''q)dt, & db'' &= (c''p - a''r)dt, & dc'' &= (a''q - b''p)dt, \end{aligned}\right\} (c').$$

and hence we have

$$pda + qdb + rdc = 0,\ pda' + qdb' + rdc' = 0,\ pda'' + qdb'' + rdc'' = 0 \quad (d'),$$

NOTE. The Title to the last Article should be "Motion of a System of Bodies round a fixed point."

We have also been obliged to defer Mr. Macully's Article on the "Summation of Trigonometrical Series." It will be inserted in the next Number.

METEOROLOGICAL OBSERVATIONS,

Made at St. Paul's College, near Flushing, L. I., for 37 successive hours, commencing at 6 A. M. of the 21st September, 1838, and ending at 6 P. M. of the following day.

Height of Barometer above low water in L. I. Sound 25 feet.
Lat. 40° 47′ 30″ *N., Long.* 73° 46′ *W. nearly.*

Hour.	Barometer Corrected.	Attached Therm'ter.	External Therm'ter.	Wet Bulb Therm'ter.	Winds —from—	Clouds —to—	Strength of wind.	REMARKS. Rain began at 7½ A. M.; continued to 2½ P. M.; ½ inch of rain fell.
6	30.104	63	62	61½	NE	SW	Gentle.	Stratus Clouds and mist.
7	.098	63	62½	62	"	"	"	"
8	.093	64	63	62½	"	"	"	Small rain and mist.
9	.091	64	64	63½	"	"	"	"
10	.088	65	64½	63¾	"	"	"	"
11	.077	66	66½	66	"	"	"	"
12	.071	67	67½	67	"	"	Light.	Heavy rain "
1	.071	67	68	67½	"	"	"	Light "
2	.054	68½	69½	69	"	"	"	" "
3	.037	69	71	70	SW	N	"	Clouds breaking.
4	.037	69	70	69½	SE	NW	Gentle.	Stratus Clouds.
5	.033	68½	68½	68	"	"	"	"
6	.032	68	67	6[illegible]	"	"	Light.	"
7	.029	68	67	66	"	"	"	"
8	.029	68	67	66	"	"	"	"
9	.032	68	67	66	"	"	Fresh.	"
10	.030	68	67	66	"	"	Brisk.	"
11	.024	68	67	66	"	"	"	"
12	.017	68	67	66	"	"	High.	"
1	.007	68	67½	66½	"	"	Brisk.	"
2	29.998	68	67½	66½	"	"	"	"
3	.973	69	67½	67	"	"	"	"
4	.971	69	67½	67	"	"	Fresh.	"
5	.965	68	67½	67	"	"	"	"
6	.963	68	67	67	"	"	"	"
7	.958	70	69¾	69	"	"	"	"
8	.954	70	70½	70	"	"	"	"
9	.946	72	76½	74	"	N	"	" breaking in E.
10	.943	73	76	74	"	"	"	"
11	.941	73	75	73	"	"	Gentle.	Cumuli.
12	.930	75	77	74½	"	"	"	"
1	.922	75	78	75	S	"	"	"
2	.918	76	78½	75	"	"	"	"
3	.918	75	76	72	"	"	"	Mostly clear.
4	.913	74	72½	70	"		"	Clear.
5	.913	73	71	68½	"		"	"
6	.912	72	69	67	"		"	"
	30.003	69	69	67¾		Means.		

METEOROLOGICAL OBSERVATIONS,

Made at St. Paul's College, near Flushing, L. I., for 37 successive hours, commencing at 6 A. M. of the 21st December, 1838, and ending at 6 P. M. of the following day.

Height of Barometer above low water in L. I. Sound, 25 feet.
Lat. 40° 47′ 30″ N., Long. 73° 46′ W. nearly.

Hour.	Barometer Corrected.	Attached Therm'ter.	External Therm'ter.	Wet Bulb Therm'ter.	Winds —from—	Clouds —to—	Strength of Wind.	REMARKS.
6	29.644	48	28	27	S. W.	N. E.	Gentle.	Stratus Clouds.
7	.642	48	28½	27½	"	"	"	"
8	.640	48	28	27½	"	"	"	"
9	.644	48	30	29½	"	"	"	"
10	.656	52	31½	31	"	"	"	"
11	.642	57	33	32	"	"	"	"
12	.620	67	33¼	32½	"	"	"	Cirrus Clouds.
1	.608	70	33½	32½	"	E.	"	"
2	.608	66	33	31½	"	"	"	"
3	.614	61	32½	31	"	"	"	"
4	.620	61	32	30½	"	"	"	Mostly clear.
5	.610	68	30	29	"	"	"	Thin Stratus Clouds.
6	.616	68	30	29	"	"	"	"
7	.612	69	29	28½	"	"	Light.	Thin misty clouds to the W.
8	.626	67	28	27½	"	"	"	[the stars visible through them.
9	.628	64	29	28½	"	"	"	"
10	.632	64	29	28½	"	"	"	"
11	.618	67	30	29	W.	"	"	Clouds darker.
12	.606	68	30½	29½	N.		"	"
1	.607	66	30½	29½	"		"	"
2	.595	65	31	30			Calm.	"
3	.567	62	32	31			"	"
4	.537	59	34	32½	E.		Light.	Bank of thin stratus clouds.
5	.531	58	34½	32½	"		"	"
6	.508	57	37	34	"	S. W.	Gentle.	"
7	.506	52	35	33½		"	Calm.	"
8	.496	53	35	34		N. W.	"	Cumulo—stratus clouds.
9	.491	50	36½	36		N. E.	"	"
10	.484	52	37	36½		"	"	"
11	.484	49	41½	41		"	"	"
12	.478	49	40	39½	WSW.	"	Light.	"
1	.457	51	38½	38		"	Calm.	"
2	.454	57	38½	38	S. W.	"	Light.	Clouds lighter.
3	.453	62	38	37	"	"	"	Cirrus clouds.
4	.480	68	36¼	35	"	"	"	"
5	.493	69	32¼	31	"		"	Mostly clear.
6	.532	52	30½	29	"		"	"
	29.568	59¼	32¾	31¾	Means.			

METEOROLOGICAL OBSERVATIONS,

Made at St. Paul's College, near Flushing, L. I., for 37 successive hours, commencing at 6 A. M. of the 21st March, 1839, and ending at 6 P. M. of the following day.

Height of Barometer above low water in L. I. Sound, 25 feet.
Lat. 40° 47′ 30″ N., Long. 73° 46′ W. nearly.

Hour.	Barometer Corrected.	Attached Therm'ter.	External Therm'ter.	Wet Bulb Therm'ter.	Winds —from—	Clouds —to—	Strength of wind.	REMARKS. Rain began at 5 P. M. of the 20th, and ended at 1½ A. M. of the 22d.
6	30.056	55	34	33½	N. E.	S. W	Brisk.	Stratus Clouds—small rain.
7	.038	55	34¾	34	"	"	"	"
8	.023	55	36	35¼	"	"	"	"
9	.016	56	35½	35	"	"	"	"
10	.011	58	36	35½	"	"	"	"
11	29.974	60	36	35½	"	"	"	"
12	.940	58	37	36½	"	"	"	"
1	.903	58	37½	37	"	"	"	"
2	.864	59	37½	37	"	"	"	"
3	.839	60	37	36½	"	"	"	"
4	.833	60	36½	36	"	"	High.	"
5	.833	60	36½	36	"	"	"	"
6	.822	58	36	35½	"	"	"	"
7	.820	56	36	35½	"	"	"	"
8	.812	56	36	35½	"	"	"	"
9	.804	56	36	35½	"	"	"	"
10	.787	54	36	35½	"	"	Brisk.	"
11	.775	52	36	35½	"	"	"	"
12	.767	51	36	35½	"	"	"	"
1	.752	50	36½	36	"	"	Gentle.	"
2	.744	49	36¾	36	"	"	"	Stratus Clouds.
3	.733	50	36¾	36	"	"	"	"
4	.735	49	36¾	36	"	"	"	"
5	.733	50	36¾	36	N. W.	S.	"	"
6	.772	50	37½	36½	W.	E.	"	"
7	.779	50	39½	39	S. W.		"	Clear.
8	.773	52	41	40	W.		Brisk.	"
9	.785	52	43	42	"	E.	Light.	Few Cirrus Clouds.
10	.776	58	47½	46	WNW	"	"	"
11	.767	64	53½	46½	"	"	Brisk.	Cumulus Clouds.
12	.758	65	56	49	W	"	"	"
1	.747	64	53½	47½	"	"	"	"
2	.738	62	56	49½	"	"	"	"
3	.738	62	56	50	"		Gentle.	Cirrus Clouds.
4	.743	61	53½	48	"		"	"
5	.746	60	48½	44½	"		Light.	Clear.
6	.752	59	44½	41½	"		"	Few Cirrus Clouds.
	29.824	56½	43¼	41½		Means.		

GILL

THE

MATHEMATICAL MISCELLANY.

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JUNIOR DEPARTMENT.

ARTICLE IV.

HINTS TO YOUNG STUDENTS.—No. VII.

35. An equation, not identical, among several quantities expresses a *relation* among these quantities; it makes any one of these quantities dependent on the others for its value, and it is thence called a *function* of these quantities. Thus, it is shown in Geometry that the three sides of a right triangle are connected by the relation

$$(12.) \quad . \quad . \quad . \quad . \quad a^2 + b^2 = h^2,$$

h being the hypothenuse of the triangle; and here a is a function of b and h, inasmuch as its value is dependent upon theirs; or b is a function of a and h, or h of a and b. If one of these quantities, as h, be considered constant, then the equation expresses the relation between the two legs of all right triangles that have this hypothenuse, and either of them is a function of the other.

So if there be, among the m quantities x, y, z, &c., n independent relations, expressed analytically by so many equations, then any n of these quantities are dependent upon the other $m-n$ quantities for their values, or they are functions of these quantities. If $m = n$, the quantities are absolutely determinable from the equations, and therefore cannot vary.

36. When one or more quantities vary, any function of them varies also; the quantity and mode of this variation being dependent on the nature of the function. This kind of dependent variation is the first idea the student has to master in applying his analysis to any practicable purpose; and it will greatly facilitate his progress if he can acquire some knowledge of it from the analytical relations alone. Thus let a function a of b, be

$$a = 2b - 3;$$

then by imagining b to vary through all magnitudes, from $-\infty$ to $+\infty$, a will also vary through all magnitudes, from $-\infty$ to $+\infty$; this will be seen by substituting successively greater values for b, such as

$$-100, -10, -1, 0, 1, 10\ 100, \&c.$$

and the corresponding values of a will be

$$-203, \ -23, -5, -3, -1, \ 17, 197, \ \&c.$$

It is seen that a varies twice as fast as b does, and by taking values of b that have yet smaller differences, the mind can easily acquire the idea of the variation of both b and a through magnitudes having an insensible difference.

A variable quantity, in changing its sign, must pass either through zero or infinity; thus, while a varies by insensible differences between 1 and 10, b varies also by insensible differences between -1 and 17, and must during this variation pass through zero; this will evidently be when

$$2b-3=0, \text{ or } b=1\tfrac{1}{2}.$$

The converse, however, does not follow. That is, a variable function in passing through zero or infinity, does not necessarily change its sign; thus if the function a were

$$a=(2b-3)^2,$$

it could never change its sign, but while b varies from $-\infty$ to $+\infty$, a will decrease from $+\infty$ to 0, which value it would have when $b=1\frac{1}{2}$, and it afterwards increases again to $+\infty$.

37. If, in equation (12), h be considered constant, and a a function of b, the limiting values of b are $-h$ and $+h$; for if b were $< -h$ or $> +h$, a^2 would be < 0, and therefore a imaginary. Making then, b vary from $-h$ to $+h$, there will be, for every value of b, two values of a, equal to each other, but with contrary signs, expressed by the forms

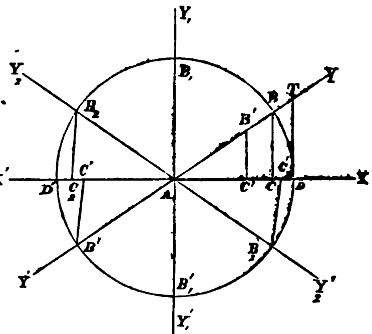
$$a=+\sqrt{h^2-b^2}, \ a=-\sqrt{h^2-b^2},$$

these values increase numerically, from 0 when $b=-h$, to h when $b=0$, and then decrease again to 0 when $b=+h$.

On the contrary, if b were considered constant, and a a function of h, the limiting values of h would be $-b$ and $+b$; for if h were between these limits a^2 would be < 0, and a imaginary. Making then h increase from $-\infty$ to $-b$, the two values of a, expressed as before, decrease numerically from ∞ to 0; and while h increases from b to ∞, the values of a increase from 0 to ∞.

The whole theory of the variation of variables and their functions, constitutes the science called the Differential Calculus; all that the student can do in the present state of his progress, when he is supposed to have acquired a pretty correct knowledge of the direct processes of Algebra, is to gain the idea of the variation of a magnitude between two fixed or finite limits, by successive, but insensibly small increments, and the consequent variation of magnitudes that depend upon it for their values by laws expressed in algebraic equations. A very few simple examples, like the above, which may be selected at pleasure, will familiarize the idea, especially if he calls to his aid some elementary ideas of motion, as I shall presently show.

38. "The *position* of a point," says Prof. Peirce, in his excellent treatise on Geometry, "is determined by its *Distance* and *Direction* from any known point." It may also be determined, on a plane by its distances from two fixed lines on that plane, or in space by its distances from three fixed planes. If A be the known point, the distance



of the point B from A is the length of the part AB, of the indefinite straight line Y'AY drawn through these two points, intercepted between them, and expressed in known units, as yards, or feet, &c. In order to distinguish on which *side* of the fixed point A the point B is situated, the distances on one side of A are marked with the sign +, and those on the other by the sign —; thus the distance of a point from a fixed point may vary from $-\infty$ to $+\infty$, and it is zero when the point coincides with A. The idea of this variation may be acquired by imagining the *progression* of the point B along the line Y'AY, its motion beginning at an infinite distance from A on the negative side AY', and moving through A to an infinite distance from A on the positive side AY. Two consecutive positions of the point in its motion are infinitely near each other; and this minute distance is the *infinitely small increment* of the varying distance. The distance of a point B from a fixed line AX, is the length of a perpendicular BC, drawn from the point to the line; these distances being marked + when the point is on one side of the line, and — when it is on the other.

The direction of the point B from A on a plane, is determined in reference to a known or *fixed* direction, such as that of the line AX, which is called the *Angular axis*, and the difference of direction of the two straight lines AY and AX is called an *angle*. The variation of angular magnitude may be conceived by imagining the indefinite line AY to revolve about the fixed point A, beginning from coincidence with AX the angular axis, in which position the angle A is zero, since the lines have then no difference of direction. As the revolution of the line continues, the angle A increases by insensible differences. When AY becomes perpendicular to AX, $A = 90^\circ = \frac{1}{2}\pi$; when AY takes the position AX', which is the prolongation of AX, $A = 180^\circ = \pi$; when AY becomes again perpendicular to AX, or takes the position AY', $A = \frac{3}{2}\pi$; and when it coincides again with AX, $A = 360^\circ = 2\pi$. By continuing the revolution, the line passes over the same positions as before; but the angles are 360° greater in any position than during the first revolution, and thus the angle may be conceived to increase from 0° to ∞, by an indefinite number of revolutions. Negative angles are counted in the opposite direction from the angular axis, and correspond to a rotation of AY about A, from left to right, instead of from right to left as before, varying in the same manner from 0° to $-\infty$. Thus if the angle $YAX = \varphi$, φ being $< 360^\circ$, the position of the line AY, is determined either from the positive value $n \cdot 2\pi + \varphi$, or the negative value $-(n+1)\, 2\pi + \varphi$, n representing the number of complete revolutions that the line has made.

39. If from different points, B, B', &c. of the line AY, perpendiculars be let fall upon the angular axis AX, they will form with AY and AX right angled triangles ABC, AB' C', &c., which have a common angle A; they

are therefore similar, and the ratios of all their homologous sides is the same: these ratios are therefore functions of the angle A, and enable us to connect angular with linear magnitude. If the hypothenuse of any one of these right triangles, as ABC, be represented by h, and the two sides, BC by a, and AC by b; these three sides have six different ratios, which are the six *trigonometrical functions* of the angle A, and are named thus:—

(13). $\frac{a}{h} = \sin. A$, or the *sine* of the angle A,

(14). $\frac{b}{h} = \cos. A$, or the *cosine* of the angle A,

(15). $\frac{a}{b} = \tan. A$, or the *tangent* of the angle A,

(16). $\frac{h}{a} = \text{cosec}. A$, or the *cosecant* of the angle A,

(17). $\frac{h}{b} = \sec. A$, or the *secant* of the angle A,

(18). $\frac{b}{a} = \cot. A$, or the *cotangent* of the angle A.

These six functions are not independent of each other, since three quantities have but two independent ratios, and these three quantities are also related to each other as in equation (12); it follows that any five of these ratios is dependent upon the sixth. To find the equations which express their relations, multiply (13), (14), (15), severally by (16), (17), (18), then

$$(19).\qquad \begin{cases} 1 = \sin A \operatorname{cosec} A, \\ 1 = \cos A \sec A, \\ 1 = \tan A \cot A; \end{cases}$$

Divide (13) by (14), member by member, and

$$(20).\quad \frac{a}{b} = \tan A = \frac{\sin A}{\cos A},$$

consequently,

$$(21).\qquad \cot A = \frac{1}{\tan A} = \frac{\cos A}{\sin A}.$$

Divide the terms of equation (12), by h^2, b^2, a^2, severally;

$$\frac{a^2}{h^2} + \frac{b^2}{h^2} = 1,$$

$$\frac{a^2}{b^2} + 1 = \frac{h^2}{b^2},$$

$$1 + \frac{b^2}{a^2} = \frac{h^2}{a^2},$$

or writing the proper functions of the angle for these ratios,

$$(22).\quad \begin{cases} \sin^2 A + \cos^2 A = 1, \\ \tan^2 A + 1 \quad = \sec^2 A, \\ \cot^2 A + 1 \quad = \operatorname{cosec}^2 A. \end{cases}$$

The student should be able to express any one of these six functions of the angle A in terms of any other one. For instance, in terms of the *sine:*

$$\cos A = \sqrt{1 - \sin^2 A}$$

$$\tan A = \frac{\sin A}{\cos A} = \frac{\sin A}{\sqrt{1-\sin^2 A}},$$

$$\cot A = \frac{\cos A}{\sin A} = \frac{\sqrt{1-\sin^2 A}}{\sin A},$$

$$\operatorname{cosec} A = \frac{1}{\sin A},$$

$$\sec A = \frac{1}{\cos A} = \frac{1}{\sqrt{1-\sin^2 A}}.$$

40. From the relation between the sine and cosine of an angle in the first of (22), and from the discussion of a similar one in § 37, it is evident that the sine and cosine vary only between the limits -1 and $+1$. But these quantities are functions of the angle, and therefore must vary with the angle; the dependency will be more immediately seen by finding a system of straight lines, proportionable to them. If, with centre A and radius = AB = AD = the linear unit, a circle be described, and from the point B, where the circumference intersects the line AY in any of its positions, the perpendicular BC be let fall on the angular axis, and at D the line DT be drawn tangent to the circle and intersecting AY in T, we have from the definitions in (13, (14) . . . . ,

$$(23), \quad \left\{\begin{array}{l} \sin A = \frac{BC}{AB} = \frac{BC}{1} = BC, \\ \cos A = \frac{AC}{AB} = \frac{AC}{1} = AC, \\ \tan A = \frac{DT}{AD} = \frac{DT}{1} = DT, \\ \sec A = \frac{AT}{AD} = \frac{AT}{1} = AT. \end{array}\right.$$

These lines are therefore numerically represented by the same quantities as the functions to which they stand opposite, and are proportional, to them; but it must be recollected that these lines are expressed in denominate numbers, and are in fact linear magnitudes, while the trigonometrical functions are ratios of lines, or abstract numbers; thus, if radius AB $= 1$ yard and BC $= \frac{2}{3}$ yard, we have also $\sin A = \frac{\frac{2}{3}\text{ yard}}{1\text{ yard}} = \frac{2}{3}$.

Now while the line AY revolves about the point A, the point of intersection B, moves from coincidence with D round the circumference, passing through BB, when $A = \frac{1}{2}\pi$, through D′ when $A = \pi$, through BB′, when $A = \frac{3}{2}\pi$, through D when $A = 2\pi$, and similarly for the successive revolutions. The line BC $= \sin A$ is the *distance* of the point B from the angular axis, and is therefore $+$ when B is above AX, and $-$ when it is below it. This settled, it is at once seen that

When A $= 0$, sin A $= 0$;
While A varies from 0 to $\frac{1}{2}\pi$, sin A increases from 0 to 1;
When A $= \frac{1}{2}\pi$, sin A $= 1$;
While A varies from $\frac{1}{2}\pi$ to π, sin A decreases from 1 to 0;
When A $= \pi$, sin A $= 0$;
While A varies from π to $\frac{3}{2}\pi$, sin A decreases from 0 to -1;
When A $= \frac{3}{2}\pi$, sin A $= -1$;
While A varies from $\frac{3}{2}\pi$ to 2π, sin A increases from -1 to 0;
When A $= 2\pi$, sin A $= 0$;

and so on for successive revolutions. It follows that, if φ be any angle, and n any integer.

$$(24)\quad\begin{cases}\sin\varphi > 0, \text{ when } \varphi > 2n\pi \text{ and } \varphi < (2n+1)\pi;\\ \sin\varphi < 0, \text{ when } \varphi > (2n-1)\pi \text{ and } \varphi < 2n\pi;\\ \sin\varphi = 0, \text{ when } \varphi = n\pi;\\ \sin\varphi = 1, \text{ when } \varphi = (2n+\frac{1}{2})\pi;\\ \sin\varphi = -1, \text{ when } \varphi = (2n-\frac{1}{2})\pi;\end{cases}$$

Similarly, the line AC $=$ cos A is the distance from the foot of the sine to the centre, and is therefore $+$ when C is to the right of A, and $-$ when it is to the left; hence

When A $= 0$, cos A $= 1$;
While A varies from 0 to $\frac{1}{2}\pi$, cos A decreases from 1 to 0;
When A $= \frac{1}{2}\pi$, cos A $= 0$;
While A varies from $\frac{1}{2}\pi$ to π, cos A decreases from 0 to -1
When A $= \pi$, cos A $= -1$;
While A varies from π to $\frac{3}{2}\pi$, cos A increases from -1 to 0;
When A $= \frac{3}{2}\pi$, cos A $= 0$;
While A varies from $\frac{3}{2}\pi$ to 2π, cos A increases from 0 to 1;
When A $= 2\pi$, cos A $= 1$;

and so on for successive revolutions; hence

$$(25)\quad\begin{cases}\cos\varphi > 0, \text{ when } \varphi > (2n-\frac{1}{2})\pi \text{ and } \varphi < (2n+\frac{1}{2})\pi;\\ \cos\varphi < 0, \text{ when } \varphi > (2n+\frac{1}{2})\pi \text{ and } \varphi < (2n+\frac{3}{2})\pi;\\ \cos\varphi = 0, \text{ when } \varphi = (n+\frac{1}{2})\pi;\\ \cos\varphi = 1, \text{ when } \varphi = 2n\pi;\\ \cos\varphi = -1, \text{ when } \varphi = (2n+1)\pi.\end{cases}$$

From the variation of the line DT $=$ tan A, or from (20), (24), and (25);

$$(26)\quad\begin{cases}\tan\varphi > 0, & \text{when } \varphi > n\pi \text{ and } \varphi < (n+\frac{1}{2})\pi;\\ \tan\varphi < 0, & \text{when } \varphi > (n-\frac{1}{2})\pi \text{ and } \varphi < n\pi;\\ \tan\varphi = 0. & \text{when } \varphi = n\pi;\\ \tan\varphi = +\infty, & \text{when } \varphi = (2n+\frac{1}{2})\pi;\\ \tan\varphi = -\infty, & \text{when } \varphi = (2n-\frac{1}{2})\pi;\end{cases}$$

By (19) it is apparent that the cosecant has the same sign as the sine, the secant as the cosine, and the cotangent as the tangent.

41. If k be a given number, positive or negative, which is either equal to -1 or $+1$, or is comprehended between these limits; there is some angle φ, either equal to $-\frac{1}{2}\pi$ or $+\frac{1}{2}\pi$, or is comprehended between these limits, and which may be found from the inspection of a table of sines, such that

$$\sin\varphi = k;$$

but there are an infinite number of other angles which have the same sine, and if any one of these angles be represented by θ, so that

$$\sin\theta = \sin\varphi\,;$$
$$\text{then, } \sin\theta - \sin\varphi = 0,$$
$$\text{or, } 2\cos\tfrac{1}{2}(\theta+\varphi)\sin\tfrac{1}{2}(\theta-\varphi) = 0\,;$$

which is satisfied by making, either

$$\cos\tfrac{1}{2}(\theta+\varphi) = 0, \text{ and by (25), } \tfrac{1}{2}(\theta+\varphi) = (n+\tfrac{1}{2})\pi\,;$$
$$\text{or, } \sin\tfrac{1}{2}(\theta-\varphi) = 0, \text{ and by (24), } \tfrac{1}{2}(\theta-\varphi) = n\pi\,; \text{ hence}$$
$$(27), \quad \theta = 2n\pi + \varphi, \text{ or } = (2n+1)\pi - \varphi,$$

where n is any integer. Similarly there is an angle φ, either equal to π or to 0 or is comprehended between these limits, as may be found from the tables, such that

$$\cos\varphi = k\,;$$

and if θ be any other angle having the same cosine, or such that

$$\cos\theta = \varphi\,;$$
$$\text{then } \cos\theta - \cos\varphi = 0,$$
$$\text{or } 2\sin\tfrac{1}{2}(\theta+\varphi)\sin\tfrac{1}{2}(\theta-\varphi) = 0\,;$$

and therefore, either

$$\sin\tfrac{1}{2}(\theta+\varphi) = 0, \text{ and by (24), } \tfrac{1}{2}(\theta+\varphi) = n\pi,$$
$$\text{or } \sin\tfrac{1}{2}(\theta-\varphi) = 0, \text{ and by (24), } \tfrac{1}{2}(\theta-\varphi) = n\pi\,; \text{ and}$$
$$(28) \quad \theta = 2n\pi \pm \varphi,$$

n being any integer. Again, if k be any number whatever, positive or negative, an angle, φ, may be found from the tables, either equal to $-\tfrac{1}{2}\pi$ or $+\tfrac{1}{2}\pi$, or is comprehended between these limits, such that

$$\tan\varphi = k,$$

and if θ be any other angle having the same tangent, so that

$$\tan\theta = \tan\varphi\,;$$
$$\text{then } \tan\theta - \tan\varphi = 0,$$
$$\text{or } \frac{\sin(\theta-\varphi)}{\cos\theta\cos\varphi} = 0,$$
$$\text{or } \sin(\theta-\varphi) = 0\,;$$
$$\text{and, by (24), } \quad \theta - \varphi = n\pi, \text{ hence}$$
$$(29) \quad \theta = n\pi + \varphi,$$

where n is any integer. Equations (27), (28), (29), give all the angles which have the same sine, cosine or tangent

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ARTICLE V.

SOLUTIONS TO THE QUESTIONS PROPOSED IN NUMBER VII.

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(43). QUESTION I. *By* ——.

Transform the numbers 25 and 389 into a system of notation whose base, or scale of relation is 3 ; multiply them in that state, exhibiting the process, and transform the result to the decimal scale.

SOLUTION. *By Mr. W. B. Benedict, Upperville, Va.*

It is obvious that, whatever be the scale of relation of the particular numbers employed, in adding, multiplying and dividing, we must divide

by the radix, setting down the excess. By the process explained in my solution to question (32).

$$\begin{array}{rl} 389 = 112102, & \text{in the ternary scale,} \\ 25 = \quad 221 & \quad " \\ \hline 112102 & \\ 1001211 & \\ 1001211 & \\ \hline 111100012 & \text{their product;} \end{array}$$

and $111100012 = 1.3^8 + 1.3^7 + 1.3^6 + 1.3^5 + 1.3 + 2 = 9725.$

(44.) Question IV. *By* ——.

Prove that

$$(n+1)(n+2)\ldots\ldots(n+n) = 2^n \times 1.3.5\ldots\ldots(2n-1).$$

Solution. *By Mr. Alfred Birdsall, Clinton Liberal Institute.*

We have
$$(n+1)(n+2)\ldots\ldots 2n = \frac{1.2.3\ldots.n}{1.2.3\ldots.n} \times (n+1)(n+2)\ldots.2n$$
$$= \frac{1.2.3\ldots 2n}{1.2.3\ldots.n}$$
$$= \frac{2.4.6\ldots.2n \times 1.3.5\ldots.(2n-1)}{1.2.3\ldots.n}$$
$$= \frac{2^n \times 1.2.3\ldots n \times 1.3.5\ldots(2n-1)}{1.2.3\ldots\ldots n}$$
$$= 2^n \times 1.3.5\ldots.(2n-1).$$

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(45). Question III. *By L., Murray Co., Geo.*

To find x, y, z, there are given the three equations

(1), $xz = y^2$,

(2), $2xz - x - y - z = 1$,

(3), $x^2 + y^2 + z^2 = 3x + 3y + 3z$.

Solution. *By Mr. Daniel Kirkwood, York, Pa.*

We have, from (1), $y = \sqrt{xz}$, and by substitution

(4), $2xz - x - \sqrt{xz} - z = 1$,

(5), $x^2 + xz + z^2 = 3(x + \sqrt{xz} + z)$;

dividing (5) by $x + \sqrt{xz} + z$,

(6), $x - \sqrt{xz} + z = 3$,

by adding (4) and (6), $xz - \sqrt{xz} = 2$;

hence, $\sqrt{xz} = 2$ or -1;

then $x - 2\sqrt{xz} + z = 1$ or 4;

$x + 2\sqrt{xz} + z = 9$ or 0;

$\sqrt{x} - \sqrt{z} = \pm 1$ or ± 2;

$\sqrt{x} + \sqrt{z} = \pm 3$ or 0;

$\sqrt{x} = \pm 2$ or ± 1,

$\sqrt{z} = \pm 1$ or ± 1,

$x = 4$ or 1,

$z = 1$ or 1,

$y = \sqrt{xz} = \pm 2$ or ± 1.

—— There are also imaginary roots found from the equation

$$x + \sqrt{xz} + z = 0.$$

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(46). QUESTION IV. *By Mr. E. H. Delafield.*

Find what relation must have place among the co-efficients of the equation

$$x^4 + \mathrm{A}x^3 + \mathrm{B}x^2 + \mathrm{C}x + \mathrm{D} = 0,$$

so that the process for taking away its second term, may also take away its fourth term at the same time.

SOLUTION. *By Mr. J. K. Anderson, St. Paul's College.*

Let $x = y + a$, and the equation becomes

$$\begin{array}{l|l|l|l} y^4 + 4a & y^3 + 6a^2 & y^2 + 4a^3 & y + a^4 = 0, \\ \quad + \mathrm{A} & \quad + 3\mathrm{A}a & \quad + 3\mathrm{A}a^2 & \quad + \mathrm{A}a^3 \\ & \quad + \mathrm{B} & \quad + 2\mathrm{B}a & \quad + \mathrm{B}a^2 \\ & & \quad + \mathrm{C} & \quad + \mathrm{C}a \\ & & & \quad + \mathrm{D} \end{array}$$

and, in order that the second and fourth terms may be taken away by the same operation, we must have

$$4a + \mathrm{A} = 0,$$

$$4a^3 + 3\mathrm{A}a^2 + 2\mathrm{B}a + \mathrm{C} = 0,$$

and by eliminating a, we find

$$\mathrm{A}^3 - 4\mathrm{AB} + 8\mathrm{C} = 0;$$

this is the same relation as in question (36), page 17; and therefore when this relation has place among the co-efficients of an equation of the fourth degree, it can be put into any one of the three forms

$$(x^2 + ax)^2 + b(x^2 + ax) + c = 0,$$

$$(x^2 + a'x + b')^2 + c' = 0,$$

$$(x - a'')^4 + b''(x - a'')^2 + c'' = 0.$$

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(47). QUESTION V. *By ——.*

Let $\mathrm{x}_0 = 1,\ 2\mathrm{x}_1 = x + \frac{1}{x},\ 2\mathrm{x}_2 = x^2 + \frac{1}{x^2},\ 2\mathrm{x}^3 = x^3 + \frac{1}{x^3}$, &c.;

prove that

$$\mathrm{x}_{n+1} + \mathrm{x}_{n-1} = 2\mathrm{x}_n\mathrm{x}_1.$$

SOLUTION. *By Mr. W. B. Benedict.*

By the notation, $2\mathrm{x}_{n+1} = x^{n+1} + x^{-n-1}$

$$2\mathrm{x}_{n-1} = x^{n-1} + x^{-n+1}$$

then

$$\begin{aligned} 2\mathrm{x}_{n+1} + 2\mathrm{x}_{n+1} &= x^{n+1} + x^{-n-1} + x^{n-1} + x^{-n+1} \\ &= x^n . x + x^{-n} . x^{-1} + x^n . x^{-1} + x^{-n} . x \\ &= (x^n + x^{-n})\, x + (x^{-n} + x^n)\, x^{-1} \\ &= (x^n + x^{-n})\,(x + x^{-1}) \\ &= 2\mathrm{x}_n . 2\mathrm{x}_1 \end{aligned}$$

and $\mathrm{x}_{n+1} + \mathrm{x}_{n-1} = 2\mathrm{x}_n\mathrm{x}_1.$

(48). QUESTION VI. *By* ——.

Adapt the relations of the sides and angles of a plane triangle to the case where the sides are in arithmetical progression, and find the area of the triangle.

SOLUTION. *By Mr. J. K. Anderson.*

When the sides are in arithmetical progression,

$$a - b = b - c, \text{ or}$$

$$(1) \qquad 2b = a + c.$$

The general relation, $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C} = k$, being written in (1) gives

$$(2) \qquad 2\sin B = \sin A + \sin C;$$

or the sines of the angles are in arithmetical progression: eliminating any one of the three angles between this and the relation.

$$A + B + C = \pi,$$

$$(3) \qquad \begin{cases} 2\cos\tfrac{1}{2}(A + C) = \cos\tfrac{1}{2}(A - C), \\ 2\sin\tfrac{1}{2}B = \sin(A + \tfrac{1}{2}B) = \sin(C + \tfrac{1}{2}B), \end{cases}$$

and from these, if one angle be given, the others can be found. If (1) be written in the general relation

$$b^2 = a^2 + c^2 - 2ac\cos B.$$

it becomes

$$(4) \qquad 3b^2 = 4ac\cos^2\tfrac{1}{2}B;$$

and for the area,

$$(5) \qquad S = \tfrac{1}{2}ac\sin B = \frac{3b^2 \sin B}{8\cos^2\tfrac{1}{2}B}$$

$$= \tfrac{3}{4}b^2\tan\tfrac{1}{2}B.$$

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(49). QUESTION VII. *By* β.

Prove that, θ being any angle,

$$\operatorname{cosec}\theta - 2\operatorname{cosec}3\theta = \cot 3\theta \sec\theta.$$

SOLUTION. *By Mr. J. V. Campbell, St. Paul's College.*

Using the known formulas

$$\sin 3\theta = \sin\theta\,(2\cos 2\theta + 1),$$

$$\cos 3\theta = \cos\theta\,(2\cos 2\theta - 1):$$

we have

$$\operatorname{cosec}\theta - 2\operatorname{cosec}3\theta = \frac{1}{\sin\theta} - \frac{2}{\sin 3\theta}$$

$$= \frac{2\cos 2\theta + 1}{\sin 3\theta} - \frac{2}{\sin 3\theta}$$

$$= \frac{2\cos 2\theta - 1}{\sin 3\theta}$$

$$= \frac{\cos\theta\,(2\cos 2\theta - 1)}{\cos\theta\sin 3\theta}$$

$$= \frac{\cos 3\theta}{\cos\theta\sin 3\theta}$$

$$= \cot 3\theta\sec\theta$$

(50). QUESTION VIII. *By* ——.

In Navigation, find the bearing and distance from a given place on the earth's surface to another one, differing from the former 10° in latitude and 10° in longitude.

SOLUTION. *By Mr. E. H. Delafield, St Paul's College.*

Denote the distance of the two places from the north pole by d and $d + 10°$; then their middle latitude will be $90° - (d + 5°)$, and their diff. long. will be $= 10° = 600$ miles; and we have

$$\text{departure} = \text{diff. long.} \times \text{cos. mid. lat.}$$
$$= 600 + \sin (d + 5°),$$
$$\tan \text{bearing} = \frac{\text{dep.}}{\text{diff. lat.}} = \frac{600 \sin (d + 5°)}{600} = \sin (d + 5°);$$
$$\text{distance} = \text{diff. lat.} \times \text{sec. bearing}$$
$$= 600 \sqrt{1 + \sin^2 (d + 5°)}.$$

When the places are near either pole, it is necessary to use the formulas in Art. I. Vol. I. Math. Miscel., which give

$$\text{cot. course} = \frac{\text{RM}}{\text{diff. long.}} \cdot \log \left(\frac{\tan\frac{1}{2}\ (90° - l)}{\tan\frac{1}{2}\ (90° - \text{L})}\right)$$
$$= \frac{\text{RM}}{600} \cdot \log \left(\frac{\tan \frac{1}{2} d}{\tan (\frac{1}{2} d + 5°)}\right),$$
$$\text{distance} = 600 \times \text{sec. course}.$$

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(51). QUESTION IX. *By* ——.

The earth being supposed a perfect sphere, draw a great circle arc between any two points on the surface which differ from each other 10° in latitude and 10° in longitude; find its length, and the angle it makes with the meridian of either place.

SOLUTION. *By* β.

The polar distances d and $d + 10°$ will be the two sides b and a of a spherical triangle, the included angle $\text{C} = 10°$ being at the north pole; hence for the third side c, or the length of the arc drawn between them,

$$\cos c = \cos a \ \cos b + \sin a \sin b \cos \text{C}$$
$$= \cos (a - b) \cos^2 \tfrac{1}{2}\text{C} + \cos (a + b) \sin^2 \tfrac{1}{2}\text{C}$$
$$= \cos 10° \cos^2 5° + \cos (2d + 10°) \sin^2 5°;$$
$$\therefore \sin^2 \tfrac{1}{2}c = \sin^2 5° (\cos^2 5° + \sin^2 \overline{d + 5°});$$

and, from Napier's Analogies,

$$\tan\tfrac{1}{2} (\text{A} - \text{B}) = \cot\tfrac{1}{2}\text{C} \cdot \frac{\sin\frac{1}{2} (a - b)}{\sin\frac{1}{2} (a + b)} = \frac{\cos 5°}{\sin\ (d + 5°)},$$
$$\tan\tfrac{1}{2} (\text{A} + \text{B}) = \cot\tfrac{1}{2}\text{C} \cdot \frac{\cos\frac{1}{2} (a - b)}{\cos\frac{1}{2} (a + b)} = \frac{\cos^2 5°}{\sin 5° \sin (d + 5°)},$$

from which the two angles may be determined. After these are found c may be had by logarithms; for

$$\sin (d + 5°) = \cos 5° \cot \tfrac{1}{2} (\text{A} - \text{B}),$$
$$\therefore \sin^2 \tfrac{1}{2}c = \sin^2 5° (\cos^2 5° + \cos^2 5° \cot^2 \tfrac{1}{2} (\text{A} - \text{B}))$$
$$= \sin^2 5° \cos^2 5° \operatorname{cosec}^2 \tfrac{1}{2} (\text{A} - \text{B})$$

$$\text{and} \quad \sin\tfrac{1}{2}c = \frac{\sin 10^\circ}{2\sin\frac{1}{2}(\text{A} - \text{B})}$$

(52). Question X. *By* ——.

Find the points of intersection of the two ellipses

$$7y^2 + 4x^2 = 28,$$
$$6y^2 + 5x^2 = 30,$$

related to the same axes of co-ordinates, and determine the angles they make with each other at these points.

Solution. *By L., Murray Co., Geo.*

By solving the two equations for y and x, we shall have the co-ordinates of the four points of intersection,

$$y = \pm 2\sqrt{\tfrac{5}{11}}, \qquad x = \pm\sqrt{\tfrac{42}{11}};$$

let α and α' be the angles, the tangents of these curves at one of these points, make with the axis of x, and ν the angle they, or the curves, make with each other; the equations of the tangents are

$$14\sqrt{\tfrac{5}{11}}\,.\,y + 4\sqrt{\tfrac{42}{11}}\,.\,x = 28,$$
$$12\sqrt{\tfrac{5}{11}}\,.\,y + 5\sqrt{\tfrac{42}{11}}\,.\,x = 30;$$

hence $\tan\alpha = -\frac{2}{7}\sqrt{\frac{42}{5}}$, $\tan\alpha' = -\frac{5}{42}\sqrt{\frac{42}{5}}$,

and $$\tan\nu = \tan(\alpha - \alpha') = \frac{\tan\alpha - \tan\alpha'}{1 + \tan\alpha'\tan\alpha} = \tfrac{11}{84}\sqrt{\tfrac{21}{10}}$$

$$\nu = 10^\circ\, 44'\, 43''.$$

(53). Question XI. *By Mr. H. Clay.*

Find when $\varphi = 0$, the value of the expression

$$\frac{1}{\varphi^2} = \frac{1}{\tan^2\varphi}.$$

Solution. *By L.*

Since $$\tan\varphi = \varphi + \tfrac{1}{3}\varphi^3 + \tfrac{2}{15}\varphi^5 + \&c.,$$
$$\tan^2\varphi = \varphi^2 + \tfrac{2}{3}\varphi^4 + \tfrac{17}{45}\varphi^6 + \&c.$$

and $$\frac{1}{\varphi^2} - \frac{1}{\tan^2\varphi} = \frac{1}{\varphi^2} - \frac{1}{\varphi^2 + \frac{2}{3}\varphi^4 + \frac{17}{45}\varphi^6 + \&c.}$$
$$= \frac{\frac{2}{3}\varphi^2 + \frac{17}{45}\varphi^4 + \&c.}{\varphi^2 + \frac{2}{3}\varphi^4 + \frac{17}{45}\varphi^6\ \&c.}$$
$$= \frac{\frac{2}{3} + \frac{17}{45}\varphi^2 + \&c.}{1 + \frac{2}{3}\varphi^2 + \frac{17}{45}\varphi^4 + \&c.}$$

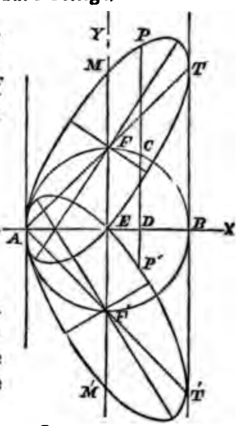
and if we now make $\varphi = 0$, we have

$$\frac{1}{\varphi^2} = \frac{1}{\tan^2\varphi}, = \tfrac{2}{3}.$$

(54). Question XII. *By* ——.

AB is the diameter of a given circle, and C any point in the circumference; from C let fall CD perpendicular to AB, and upon it take CP = AD; find the curve in which the point P is always found.

SOLUTION. *By Mr. Warren Colburn, St. Paul's College.*



Let R be the radius of the circle, the diameter AB being the axis of a, and the centre E the origin of co-ordinates. Let x, y' be the co-ordinates of the point C, and x, y those of the point P; then we have

$$y' = \sqrt{R^2 - x^2},$$

$$y = y' \pm (R + x) = \sqrt{R^2 - x^2} \pm (R + x),$$

and this equation when properly ordered becomes

$$(1) \quad y^2 + 2x^2 \mp 2yx \mp 2Ry + 2Rx = 0,$$

the upper signs being used when CP is taken on CD produced, and the lower when taken from C towards D. Using first the upper signs, the curve will necessarily pass through the origin since there is no absolute term, and if we make

$$x = R, \text{ we find } (y - 2R)^2 = 0, \; y = 2R;$$

$$\text{if } x = -R, \text{ we find } \quad y^2 = 0, \; y = 0;$$

the curve is therefore bounded by the perpendiculars to AB at A and B, touching them at A and at T, so that BT = AB. Comparing the equation (1) with the general equation of the second degree

$$(2) \quad Ay^2 + Bx^2 + 2Cxy + 2Dy + 2Ex = K,$$

we find $\quad A = 1, \; B = 2, \; C = -1, \; D = -E = -R, \; K = 0$; hence

$$AB - C^2 = 1 > 0,$$

and the curve is an ellipse, the co-ordinates of its centre being

$$k = R, \quad l = 0,$$

or the centre is at the point F where the axis of y cuts the circle; hence the line EFM = 2R is a diameter of the ellipse, and if θ be the angle which a diameter conjugate to this makes with the axis of x, we shall find $\tan\theta = 1$, and the equation of this diameter is

$$y - R = x,$$

which therefore passes through the points A and T, and the diameters AFT, EFM are conjugate to each other.

The equation of the major axis of this ellipse is

$$y - R = \tfrac{1}{2}(\sqrt{5} + 1)x,$$

which therefore cuts the axis of x at a distance

$$x = -\tfrac{1}{2}R(\sqrt{5} - 1)$$

from the origin and at an angle φ, such that

$$\tan\varphi = \tfrac{1}{2}(\sqrt{5} + 1), \text{ and } \varphi = 58^\circ\, 17'.$$

The lengths of the semiaxes, a and b, are the positive roots of the equation

$$\left(\frac{R^2}{r^2} - 1\right)\left(\frac{R^2}{r^2} - 2\right) - 1 = 0,$$

and therefore $\quad a = \tfrac{1}{2}R(\sqrt{5} + 1), \; b = \tfrac{1}{2}R(\sqrt{5} - 1),$

In a similar manner, the other locus is found to be an equal ellipse, posited in the same way on the opposite side of the axis.

—— The tangents at E to the two curves are respectively parallel to

the diameters AT, AT′ conjugate to the diameters through E, hence the curves intersect the axis of x at E in an angle of 45°, and are perpendicular to each other at this point. This is shown by Mr. Campbell, whose solution, as well as that of Mr. Deslond, are well worthy of insertion. The general discussion of equation (2), from which these particulars are deduced, will be inserted in a subsequent article.

List of Contributors to the Junior Department, and of Questions answered by each. The figures refer to the number of the questions, as marked in Number VII, Article III, page 24, Vol. II.

J. K. Anderson, Freshman Class, St. Paul's College, ans. 3, 4, 5, 6, 7.
W. B. Benedict, Upperville, Va., ans. 1, 3, 4, 5, 12.
Alfred Birdsall, Clinton Liberal Institute, ans. 1, 2, 3, 4.
β, ans. 7, 9.
J. V. Campbell, Sophomore Class, St. Paul's College, ans. 5, 7, 12.
Warren Colburn, Sophomore Class, St. Paul's College, ans. 12.
E. H. Delafield, Freshman Class, St. Paul's College, ans. 1, 3, 4, 5, 7, 8.
A. Deslond, Sophomore Class, St. Paul's College, ans. 12.
Daniel Kirkwood, York, Pa., ans. 3.
L., Murray Co, Geo., ans. all the questions.

*** The communication of "Numerator," containing correct solutions to questions 3, 4 and 7, did not come to hand until the previous part of the copy was in the printer's hands.

ARTICLE VI.

QUESTIONS TO BE ANSWERED IN NUMBER IX.

Their solutions must arrive before February 1st, 1840.

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(55). QUESTION I. *By* ——.

Prove that the product of two numbers which are, each of them, less than unity, is less than either of them.

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(56). QUESTION II. *By Numerator.*

Given that

$$(1+\sqrt{-1})^7 = 8(1-\sqrt{-1}), \text{ and } (1-\sqrt{-1})^7 = 8(1+\sqrt{-1});$$

show that one value of

$$(1+\sqrt{-1})^{\frac{1}{7}}+(1-\sqrt{-1})^{\frac{1}{7}} \text{ is } (16)^{\frac{1}{7}}.$$

(57). QUESTION III. *By* ——.

It is required to find the roots of the equation

$$x^3 - \frac{a^2 - ab + b^2}{a^{\frac{1}{2}} b^{\frac{3}{2}}} . x^2 - \frac{a^2 - ab + b^2}{b^2} . x + \left(\frac{a}{b}\right)^{\frac{3}{2}} = 0,$$

which are in geometrical progression.

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(58). QUESTION IV. *By Mr. D. Kirkwood.*

To find x, y, z, u, then are given the four equations

(1), $x^2 y^3 + uxyz + u^2 z^2 = 460$,

(2), $\sqrt{1 + 2xy - y^2} = x$,

(3), $\dfrac{u^2 z^2}{xy} + xy^2 + uz = 38\tfrac{1}{3}$,

(4), $u^3 + u^2 z^2 = 5$.

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(59). QUESTION V. *By* ——.

Find x from the equation

$$\sqrt{abx - \frac{bc}{x}} + \sqrt{ac \quad \frac{bc}{x}} = ax.$$

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(60). QUESTION VI. *By* ——.

In any plane triangle ABC, if the angle made by the line drawn from the angle C to the middle point of the side c be denoted by θ, prove that

$$2 \cot \theta = \cot A - \cot B.$$

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(61). QUESTION VII. *By Eltinge, New-Brunswick.*

It is required to find a point in a given equilateral triangle from which if straight lines, the lengths of which are denoted by k, l, m, be drawn to the three vertices of the triangle, we may have

$$\tfrac{1}{3}(k + l) + m = \tfrac{1}{4}(k + m) + l = \tfrac{1}{5}(l + m) + k.$$

---

(62). QUESTION VIII. *By* ——.

Eliminate θ between the two equations

$$x \sin \theta + a \operatorname{cosec} \theta = c,$$
$$x \cos \theta + b \sec \theta = c.$$

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(63). QUESTION IX. *By L., Murray Co., Geo.*

Prove that

$$\cot \theta - \tan \theta = 2 \cot 2\theta,$$

and thence show that

$$\tan \theta + 2 \tan 2\theta + \ldots\ldots + 2^{n-1} \tan 2^{n-1} \theta = \cot \theta - 2^n \cot 2^n \theta.$$

(64). QUESTION X. *By* ——.

Find the values of x corresponding to the *maxima* or *minima* values of the expression

$$2x^3 - 3(a+b)x^2 + 6abx + c$$

and distinguish the *maxima* from the *minima* values.

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(65). QUESTION XI. *By* ——.

Given the equation

$$(\text{R}-r)^2y^2 + \{(\text{R}-r)^2 - d^2\}x^2 - \{(\text{R}-r)^2 - d^2\}dx = \tfrac{1}{4}\{(\text{R}-r)^2 - d^2\}^2;$$

it is required to find its form when R and d become infinite, but have a given finite difference.

---

(66). QUESTION XII. *By* ——.

When the general equation of the second degree

$$\text{A}y^2 + \text{B}x^2 + 2\text{C}xy + 2\text{D}y + 2\text{E}x = \text{K},$$

represents a parabola, it is required to find its parameter, and the position of its axis, focus and vertex, in terms of the co-efficients.

SENIOR DEPARTMENT.

ARTICLE IV.

SOLUTIONS TO THE QUESTIONS PROPOSED IN NUMBER VI.

(98.) QUESTION I. *By an Engineer.*

a, b, c, d are four points on a hill which is to be reduced to a level of 10 feet below a; the surface nearly coincides with the planes drawn through a, b, c, and through a, c, d. It is required to find the quantity of earth to be removed from this part of the hill; the relative position of the points being given, as below:

Stations.	Bearing.	Distance.	Elevation.
a to b	S. 23° 17′E.	51 feet 3 in.	— 5° 25′
b to c	S. 54° 38′W.	79 " 10 "	+ 8° 37′
c to d	N. 10° 15′W.	63 " 5 "	+ 10° 9′
d to a	———	———	———

SOLUTION *By L., Murray Co., Geo.*

a', b', c', d', are the projections of a, b, c, d on a horizontal plane drawn 10 feet below a; by the process of question (85) we find

$$\begin{aligned} a'b' &= 51{,}0211, & aa' &= 10,\\ b'c' &= 78{,}9322, & bb' &= 5{,}1621,\\ c'd' &= 62{,}4242, & cc' &= 17{,}1229,\\ a'c' &= 102{,}565\,; & dd' &= 28{,}2985\,; \end{aligned}$$

hence, area of $a'b'c' = 1968,994$ square feet,
area of $a'd'c' = 1871,563$ do.

Pass a verticle plane through a, c, a', c' it will divide the solid into two truncated triangular prisms, and the

solidity of $abca'b'c' = \frac{1}{3}a'b'c'(aa' + bb' + cc') = 21189,465$ cu. ft
solidity of $adca'd'c' = \frac{1}{3}a'd'c'(aa' + dd' + cc') = 34574,85$ do.

and their sum is the whole solid $= 2065,345$ cu. yds. $= 55764,315$.

(99). Question II. *By Wm. Lenhart, Esq.*

Show how to find those integers whose cubes terminate with the three digits 048.

First Solution. *By Prof. Peirce.*

This question is to solve the equation

$$x^3 - 1000n = 48, \text{ or } x^3 \equiv 48 \text{ (mod. 1000)},$$

so that x must satisfy the congruences,

$$x^3 \equiv 48 \text{ (mod. 8)}, \quad \text{and } x^3 \equiv 48 \text{ (mod. 125)}.$$

Now the only root of the first of these inequalities is

$$x \equiv 0 \text{ (mod. 2)}, \text{ or } x = 2x',$$

in which x' is any integer. We have then to satisfy the inequality

$$8x'^3 \equiv 48 \text{ (mod. 125)}, \text{ or } x'^3 \equiv 6 \text{ (mod. 125)};$$

which involves $x' \equiv 1$ (mod. 5), or $x' = 5x'' + 1$,
and again we have $(1 + 5x'')^3 \equiv 6$ (mod. 125),
or $3x'' + 15x''^2 + 25x'''^3 \equiv 1$ (mod. 25),
which involves $3x'' \equiv 1$ (mod. 5), or $x'' = 5x''' + 2$;
so that the last congruence to be satisfied is

$$15x''' + 66 \equiv 1 \text{ (mod. 25)}, \text{ or } 3x'' + 13 \equiv 0 \text{ (mod. 5)},$$
$$\text{or } x''' + 1 \equiv 0 \text{ (mod. 5)}, \text{ whence } x''' = 4 + 5m;$$
$$x'' = 22 + 25m, x' = 111 + 125m, x = 222 + 250m;$$

and the numbers required are the arithmetical series 222, 472, &c.

Second Solution. *By Mr. Alfred Birdsall, Clinton Liberal Institute.*

Since the cube terminates with 048, it may be represented by $1000\,a + 48$, and the root evidently terminating with 2, may be represented by $10n + 2$, then

$$(10n + 2)^3 = 1000n^3 + 600\,n^2 + 120n + 8 = 1000\,a + 48,$$
$$\text{and} \quad 25n^3 + 15n^2 + 3n - 1 = 25a;$$

then we must have $3n - 1$ divisible by 5, which will be the case if $n = 2 + 5u$, then the preceding equation becomes

$$5\,(2 + 5u)^3 + 3\,(2 + 5u)^2 + 3u + 1 = 5a;$$

so that a will be an integer for all values of u that make

$$3u + 13, \text{ or } 3u + 3, \text{ or } u + 1 \text{ divisible by } 5,$$

and this will be so when $u = 5w + 4$; whence the number is

$$10n + 2 = 50u + 22 = 250w + 222.$$

If $w = 0$, the least root is 222.

——— Mr. Lenhart, the proposer, mentions that these numbers have several singular properties, one of which is, that

$$4\,(250m+222)^3-(250m+223)^3=5^3\,.\,\text{A}.$$

Mr. Perkins also notices the singular property that all numbers terminating with the figures 12890625, have all their integral powers terminating with the same figures; or that

$$(\ldots\ldots12890625)^n=\ldots\ldots\ldots12890625.$$

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(100). QUESTION III. *Generalized from Peirce's Algebra.*

n men play together on the condition that he who loses shall give to all the rest as much as they already have. They play n games, and each loses in his turn, after which it is found that they have given sums of money. How much had each when they began to play?

SOLUTION. *By Mr. E. H. Delafield, St Paul's College.*

Let x_m denote the sum the m^{th} man had when they began to play,
a_m the sum he had at the end of the n^{th} game;
and let $\quad s=a_1+a_2+\ldots+a_m\ldots\ldots+a_n$.
Then this man, who wins the m^{th} game, would have, at the end of the

$$\begin{array}{ll}
\text{1st game} & \ldots\ldots 2x_m, \\
\text{2d } `` & \ldots\ldots 2^2x_m, \\
 & \&c. \\
(m-1)^{st}\ `` & \ldots\ldots 2^{m-1}x_m, \\
m^{th}\ `` & \ldots\ldots 2^m x_m - s, \\
(m+1)^{st}\ `` & \ldots\ldots 2^{m+1}x_m-2s, \\
 & \&c. \\
(m+k)^{th}\ `` & \ldots\ldots 2^{m+k}x_m-2^k s,
\end{array}$$

Let $m+k=n$, then, at the end of the n^{th} game, he will have

$$2^n x_m-2^{n-m}s=a_m,$$
$$x_m=2^{-m}s+2^{-n}a_m.$$

——— The solution of Mr. Kirkwood was also very neat.

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(101). QUESTION IV. *By P.*

A hemisphere and cone are fastened with their equal bases together. It is required to find the height of the cone, so that the whole solid may be in equilibrium on any point of the curve surface of the hemisphere.

SOLUTION. *By Mr. M. P. Barton, Jun., Esperance, N. Y.*

It is evident that the centre of gravity of the cone and hemisphere, considered as one body of uniform density, must be in the centre of the sphere of which the hemisphere is a part. Put r = radius of the hemisphere, and x = altitude of the cone; then

$$\tfrac{2}{3}\pi r^3=\text{content of the hemisphere},$$
$$\tfrac{2}{3}\pi r^3\times\tfrac{3}{8}r=\tfrac{1}{4}\pi r^4=\text{its moment},$$
$$\tfrac{1}{3}\pi r^2x=\text{content of the cone},$$
$$\tfrac{1}{3}\pi r^2x\times\tfrac{1}{4}x=\tfrac{1}{12}\pi r^2x^2=\text{its moment}.$$

hence
$$\tfrac{1}{4}\pi r^4=\tfrac{1}{12}\pi r^2x^2$$
$$x=r\sqrt{3}$$

and the moments being equal when the axis is horizontal, they will be

equal in all positions. The whole solid may evidently be generated by the revolution of a semicircle and equilateral triangle described on different sides of the same line, about a perpendicular to this line through its middle point.

(102.) Question V. *By* ψ.

Given the equation

$$y^4 - 9y^3x + 2x^2 = 0;$$

to express y in a series of monomials, arranged 1°. according to the ascending, and 2°. according to the descending powers, of x.

Solution. *By Prof. M. Catlin, Hamilton College, Clinton.*

Let $y = \text{A}x^a + \text{B}x^b + \text{C}x^c + \&\text{c.}$, then the equation becomes

$$\begin{aligned} 0 = {} & \text{A}^4x^{4a} + 4\text{A}^3\text{B}x^{3a+b} + 6\text{A}^2\text{B}^2x^{2a+2b} + 4\text{AB}^3x^{a+3b} + \&\text{c.} \\ & - 9\text{A}^3x^{3a+1} - 27\text{A}^2\text{B}x^{2a+b+1} - 27\text{AB}^2x^{a+2b+1} - 27\text{A}^2\text{C}x^{2a+c+1} + \&\text{c.} \\ & + 2x^2 \quad - \quad - \quad - \quad - \quad - \quad - \quad - \end{aligned} \quad (1).$$

If $a = \frac{1}{2}$, then $b = 1$, $c = \frac{3}{2}$, $d = 2$, &c.; and

$$0 = \left.\begin{array}{l} \text{A}^4 \\ +2 \end{array}\right| x^2 + \left.\begin{array}{l} 4\text{A}^3\text{B} \\ -9\text{A}^3 \end{array}\right| x^{\frac{5}{2}} + \left.\begin{array}{l} 6\text{A}^2\text{B}^2 \\ +4\text{A}^3\text{C} \\ -27\text{A}^2\text{B} \end{array}\right| x^3 + \left.\begin{array}{l} 12\text{A}^2\text{BC} \\ +4\text{AB}^3 \\ +4\text{A}^3\text{D} \\ -27\text{A}^2\text{C} \\ -27\text{AB}^2 \end{array}\right| x^{\frac{7}{2}} + \&\text{c.} \quad . \quad (2);$$

hence $\text{A} = (-2)^{\frac{1}{4}}$, $\text{B} = \frac{9}{4}$, $\text{C} = \frac{3}{2} \cdot \frac{9^2}{4^2} \cdot \frac{1}{(-2)^{\frac{1}{4}}}$, $\text{D} = \frac{6\,.\,2}{2\,.\,3} \cdot \frac{9^3}{4^3} \cdot \frac{1}{(-2)^{\frac{1}{2}}}$,

$\text{E} = \frac{9\,.\,5\,.\,1}{2\,.\,3\,.\,4} \cdot \frac{9^4}{4^4} \cdot \frac{1}{(-2)^{\frac{3}{4}}}$, $\text{F} = \frac{12\,.\,8\,.\,4\,.\,0}{2\,.\,3\,.\,4\,.\,5} \cdot \frac{9^5}{4^5} \cdot \frac{1}{-2}$, &c. and

$$y = (-2)^{\frac{1}{4}}x^{\frac{1}{2}} + \tfrac{9}{4}x + \tfrac{3}{2} \cdot \frac{9^2}{4^2} \cdot \frac{1}{(-2)^{\frac{1}{4}}}x^{\frac{3}{2}} + \frac{6\,.\,2}{2\,.\,3} \cdot \frac{9^3}{4^3} \cdot \frac{1}{(-2)^{\frac{1}{2}}}x^2 + \&\text{c.} \quad (3)$$

Equation (1) may also be satisfied by putting $4a = 3a + 1$, $a = 1$;
$\therefore b = -1$, $c = -3$, $d = -5$, &c. Therefore we shall have

$$0 = \left.\begin{array}{l} \text{A}^4 \\ -9\text{A}^3 \end{array}\right| x^4 + \left.\begin{array}{l} 4\text{A}^3\text{B} \\ -18\text{A}^2\text{B} \\ -9\text{BA}^2 \\ +2 \end{array}\right| x^2 + \left.\begin{array}{l} 6\text{A}^2\text{B}^2 \\ +4\text{A}^3\text{C} \\ -27\text{AB}^2 \\ -27\text{A}^2\text{C} \end{array}\right| x^0 + \left.\begin{array}{l} 4\text{AB}^3 \\ +12\text{A}^2\text{BC} \\ +4\text{A}^3\text{D} \\ -9\text{B}^3 \\ -54\,\text{ABC} \\ -27\,\text{A}^2\text{D} \end{array}\right| x^{-2} + \&\text{c}, \quad (4);$$

Equating the co-efficients with zero, we shall obtain

$\text{A} = 9$, $\text{B} = -\frac{2}{9^3}$, $\text{C} = -\frac{3\,.\,4}{9^7}$, $\text{D} = -\frac{9\,.\,10}{2\,.\,3} \cdot \frac{2^3}{9^{11}}$, &c.

$$y = 9x - \frac{2}{9^3}x^{-1} - \frac{6}{2} \cdot \frac{2^2}{9^7}x^{-3} - \frac{9\,.\,10}{2\,.\,3} \cdot \frac{2^3}{9^{11}}x^{-5} - \frac{12\,.\,13\,.\,14}{2\,.\,3\,.\,4}\frac{2^4}{9^{15}}x^{-7} - \&\text{c.}$$

——— Mr. Perkins notices that in the first development, (3), all the terms standing in the order $4n + 2$, n being any integer, vanish from having 0 for a factor, and the terms between the vanishing terms are all alternately positive and negative.

(103). QUESTION VI. *By* ——.

It is required to place a given parabola so as to touch a given line at a given point in it, and to intersect a second given line at a given angle.

FIRST SOLUTION. *By James F. Macully, Esq. New-York.*

Let the first given line be the angular axis, the given point being the pole; the equation of the given parabola (M. Misc., Vol. I. page 12,) is

$$(1). \qquad r = \frac{p \sin \omega}{\sin \varepsilon \sin^2 (\omega - \varepsilon)},$$

p being the parameter, ε the angle the axis of the parabola makes with the angular axis. Also (Misc., Vol, II. page 29), the equation of the second given line is

$$(2). \qquad r = \mathrm{P} \sec (\omega - \alpha);$$

and the equation of a tangent to the parabola at the point $r\omega$, is

$$r_{,} \left\{ \cos (\omega_{,} - \omega) - \sin (\omega_{,} - \omega) \frac{dr}{r\, d\omega} \right\} = r.$$

If β be the angle this line makes with the angular axis, the angle $\alpha - \frac{1}{2}\pi - \beta$ is given, therefore β is given, and it is evidently such that

$$\cos (\beta - \omega) - \sin (\beta - \omega) \frac{dr}{r d\omega} = 0$$

or, since, by (1), $\quad \dfrac{dr}{rd\omega} = - \dfrac{\sin \varepsilon + \sin \omega \cos (\omega - \varepsilon)}{\sin \omega \sin (\omega - \varepsilon)}$,

this equation becomes

$$(3). \qquad \sin \omega \sin (\beta - \varepsilon) + \sin \varepsilon \sin (\beta - \omega) = 0$$

Eliminating r and ω between the equations (1), (2), (3), we have

$$(4). \quad 4\mathrm{P} \sin^2 \varepsilon \sin^2 (\beta - \varepsilon) = p \sin \beta \left\{ \sin \varepsilon \cos (\alpha + \beta) + \sin (\varepsilon - \beta) \cos \alpha \right\},$$

from which ε may be found, which determines the position of the parabola.

SECOND SOLUTION. *By Mr. B. Birdsall, Clinton Liberal Institute.*

By using the usual formulas of transformation, the equation of the parabola will be

$$\left\{ (y - \beta) \cos\theta - (x - \alpha) \sin\theta \right\}^2 = 2p \left\{ (y - \beta) \sin\theta + (x - \alpha) \cos \theta \right\} \quad (1),$$

α and β being the co-ordinates of its vertex, and θ the angle, its axis makes with the axis of x, which we will suppose to be the second given line of the question, the origin being at the intersection of the two given lines, and therefore the equation of the first is

$$y' = ax'. \quad . \quad . \quad . \quad . \quad (2).$$

This line is to be tangent to the parabola, at the given point $y'x'$, and therefore,

$$\frac{dy}{dx} = \frac{dy'}{dx'} = \frac{\left\{ (y' - \beta) \cos\theta - (x' - \alpha) \sin\theta \right\} \sin\theta + p \cos\theta}{\left\{ (y' - \beta) \cos\theta - (x' - \alpha) \sin\theta \right\} \cos\theta - p \sin\theta} = a \qquad (3).$$

This given point being in the parabola

$$\left\{ (y' - \beta) \cos\theta - (x' - \alpha) \sin\theta \right\}^2 = 2p \left\{ (y' - \beta) \sin\theta + (x' - \alpha) \cos\theta \right\} \ (4).$$

But the line is to cut the axis of x, at an angle whose tangent is b, and therefore, for this point, since $y = 0$,

$$\left\{ \beta \cos \theta + (x - \alpha) \sin \theta \right\}^2 = 2p \left\{ - \beta \sin \theta + (x - \alpha) \cos\theta \right\} \ (5),$$

$$\frac{dy}{dx} = \frac{\left\{ \beta \cos\theta + (x - \alpha) \sin\theta \right\} \sin\theta - p \cos \theta}{\left\{ \beta \cos\theta + (x - \alpha) \sin\theta \right\} \cos\theta + p \sin \theta} = b \quad . \qquad (6).$$

Equations (3), (4), (5), (6) are sufficient to determine the unknown quantities x, α, β, θ, and thence the position of the parabola.

——— The most interesting case of the question is that where (in the first solution) $\alpha = \frac{1}{2}\pi$, or the given straight lines are parallel, and also $\beta = \frac{1}{2}\pi$; then equation (4) becomes

$$\sin^3\varepsilon - \sin\varepsilon + \frac{p}{4\text{P}} = 0,$$

It follows that if $\quad 8p > 3\text{P}\sqrt{3}$

the parabola can be placed but in one position; but if

$$8p = \text{ or } < 3\text{P}\sqrt{3},$$

it can be placed in three positions, such that it touches one given line at a given point, and is perpendicular to a parallel line.

(104). QUESTION VII. *By Prof. G. R. Perkins, Utica Academy.*

Given the sum of the squares, and the sum of the fourth powers of four lines drawn from a point to the four vertices of a regular tetraedron, to find the side of the tetraedron.

SOLUTION. *By the Proposer.*

If we refer the corners of the tetraedron to three rectangular axes through its centre, the plane of xy being parallel to one of the faces, and the axis of x parallel to one of the edges of this face, we shall have the co-ordinates of the vertices

$$\begin{array}{ccc} -x, & -\frac{1}{3}x\sqrt{3}, & -\frac{1}{6}x\sqrt{6} \\ x, & -\frac{1}{3}x\sqrt{3}, & -\frac{1}{6}x\sqrt{6} \\ 0, & \frac{2}{3}x\sqrt{3}, & -\frac{1}{6}x\sqrt{6} \\ 0, & 0\ , & \frac{1}{2}x\sqrt{6} \end{array}$$

where x is half the required side of the tetraedron. Now if we represent the co-ordinates of the point by x', y', z', and the four distances of this point from the vertices by a, b, c, d, we shall find

$$(x'+x)^2+(y'+\tfrac{1}{3}x\sqrt{3})^2+(z'+\tfrac{1}{6}x\sqrt{6})^2 = a^2 \quad . \quad . \quad (1),$$
$$(x'-x)^2+(y'+\tfrac{1}{3}x\sqrt{3})^2+(z'+\tfrac{1}{6}x\sqrt{6})^2 = b^2 \quad . \quad . \quad (2),$$
$$x'^2+(y'-\tfrac{2}{3}x\sqrt{3})^2+(z'+\tfrac{1}{6}x\sqrt{6})^2 = c^2 \quad . \quad . \quad (3),$$
$$x'^2+\quad y'^2 \quad +(z'-\tfrac{1}{2}x\sqrt{6})^2 = d^2 \quad . \quad . \quad (4).$$

By eliminating x', y', z', we find

$$12x^2 = a^2+b^2+c^2+d^2 \pm 2\sqrt{(a^2+b^2+c^2+d^2)^2 - 3(a^4+b^4+c^4+d^4)} \quad (5),$$

Hence, if we put $\quad \text{s} = 2x$, the required side,

$$\text{A} = a^2+b^2+c^2+d^2,$$
$$\text{B} = a^4+b^4+c^4+d^4;$$

then $\quad \text{s} = \sqrt{\tfrac{1}{3}}\,.\,\{\text{A} \pm 2\sqrt{\text{A}^2-3\text{B}}\}^{\frac{1}{2}} \quad . \quad . \quad . \quad . \quad (6).$

If r be the radius of the sphere passing through the vertices of the tetraedron, and ϱ the distance of the point from the centre

$$r^2 = \tfrac{3}{8}\text{s}^2 = \tfrac{1}{8}\{\text{A} \pm 2\sqrt{\text{A}^2-3\text{B}}\};$$

and by adding together the equations (1), (2), (3), (4),

$$x'^2+y'^2+z'^2 = \varrho^2 = \tfrac{1}{4}(\text{A} - 6x^2) = \tfrac{1}{8}\{\text{A} \mp 2\sqrt{\text{A}^2-3\text{B}}\}.$$

Now, we have always $A > 2\sqrt{A^2 - 3B}$, because
$A^2 - 4(A^2 - 3B)$, or $3(4B - A^2)$
can be put into the form
$(3d^2 - a^2 - b^2 - c^2)^2 + 2(2c^2 - a^2 - b^2)^2 + 6(a^2 - b^2)^2$,
and is therefore > 0; it follows that there are always two tetraedrons which solve the problem, the point from which the lines are drawn for one tetraedron being any where in the surface of the sphere circumscribed about the other. The case of
$$A^2 - 3B = 0,$$
is an exception; the two tetraedrons and spheres being then confounded with each other. If $a = b = c = d$, one of the tetraedrons is reduced to a point coinciding with the centre of the other.

(105.) QUESTION VIII. *By Wm. Lenhart, Esq. York, Penn.*

It is required to find n numbers such that their sum increased by the sum of their cubes shall be equal to the sum of n other numbers increased by the sum of their cubes.

FIRST SOLUTION. *By the Proposer.*

Suppose that two numbers were required, then we should have
$$x^3 + y^3 + x + y = v^3 + w^3 + v + w \qquad (1).$$
Or, supposing $x^3 + y^3$ and $v^3 + w^3$ to be equal to tabular numbers t and t' respectively,
$$t - t' = (v + w) - (x + y) \qquad (2).$$
by which we perceive that when $t > t'$, the sum of the roots of the cubes which compose t', is greater than the sum of the roots of the cubes composing t; therefore all we have to do is to look into the table of numbers composed of two cubes, for two numbers whose difference is equal to the difference of the sums of the roots of the component cubes, as in (2), and these roots will be the numbers required. To illustrate: By Table we have $t = 344 = (7)^3 + (1)^3$, and $t' = 341 = (5)^3 + (6)^3$; and since $344 - 341 = (5 + 6) - (7 + 1)$ therefore $x = 5$, $y = 6$, and $v = 7$, $w = 1$; the numbers required. If n numbers be required then
$$x^3 + y^3 + z^3 \text{ \&c. } + x + y + z \text{ \&c.} = v^3 + w^3 + u^3 \text{ \&c. } + v + w + u \text{ \&c.} \quad (3),$$
in which it is evident that all the numbers, with the exception of two on each side of the equation, may be assumed. This being done the equation takes the form of
$$x^3 + y^3 + x + y + m = v^3 + w^3 + v + w + \eta. \quad . \quad (4),$$
in which since $x^3 + y^3$ or t is supposed to be greater than $v^3 + w^3$ or t', we suppose $\eta > m$, and thence putting $\eta - m = \eta'$, we shall have
$$t - t' = (v + w) - (x + y) + \eta' \quad . \quad . \quad (5),$$
or, if $t = t'$
$$x + y = v + w + \eta'. \quad . \quad . \quad . \quad (6),$$
which are readily answered by a simple inspection of the Table of numbers and their two component cubes. For instance: suppose three numbers were required; then assuming $z = 1$ and $u = 2$, we shall have $z^3 + z = m = 2$, $u^3 + u = \eta = 10$, $\eta - m = \eta' = 8$, and
$$t - t' = (v + w) - (x + y) + 8 \quad . \quad . \quad (7).$$

Now, by the Table,

$$t = 1853 = (5)^3 + (12)^3 \quad \text{and} \quad t' = 1843 = (8)^3 + (11)^3$$

therefore $\quad t - t' = 10 = (8 + 11) - (5 + 12) + 8$

and consequently $x = 12$, $y = 5$, $z = 1$; and $v = 11$, $w = 8$ and $u = 2$. In the same way we readily find the fcur numbers 14, 13, 11, 8; and the other four 17, 12, 5 and 3; also the five numbers 21, 14, 10, 4, 1; and the other five 20, 17, 5, 3 and 2.

We may, however, from having several sets of numbers to answer the first case of the question, find n numbers to answer independently of assuming any of the numbers. Suppose, for instance, we have

$$x^3 + y^3 + x + y = v^3 + w^3 + v + w,$$

and also $\quad x'^3 + y'^3 + x' + y' = v'^3 + w'^3 + v' + w',$

then by adding these equations together the reader will perceive that we shall succeed in obtaining four different numbers on each side: Or, if $x = v'$, or $y = w'$ or $x = v$, or $y' = w$, by cancelling either of the equal values, we shall get three different numbers on each side. And it is obvious that in the same manner n numbers may be found.

SECOND SOLUTION. *By Prof. Peirce.*

Let the m^{th} number of the first set be $\quad a_m x + b_m$,
and the m^{th} number of the second set $\quad a_m x + b_{n-m+1}$,
and we have for the sums

$$x^3 s\,.\,a^3_m + 3x^2 s\,.\,a^2_m b_m + xs\,.\,(3a_m\ b^2_m + a_m) + s\,.\,(b^3_m + b_m)$$
$$= x^3 s.\,a^3_m + 3x^2 s.\,a^2_m b_{n-m+1} + xs.\,(3a_m b^2_{n-m+1} + a_m) + s.\,(b^3_m + b_m)$$

so that
$$x = \frac{s.\,a_m\ (b^2_{n-m+1} - b^2_m)}{s.\,a^2_m\ (b_m - b_{n-m+1})},$$

whence the required numbers are found.

(106.) QUESTION IX. *By J. F. Macully, Esq. New-York.*

It is required to find the sum of the series

$$\frac{1 + 4\cos^4\theta}{\cos^2 2\theta \cos^2\theta} + \frac{1}{4^2}.\frac{1 + 4\cos^4\frac{1}{4}\theta}{\cos^2\frac{1}{2}\theta\cos^2\frac{1}{4}\theta} + \frac{1}{4^4}.\frac{1 + 4\cos^4\frac{1}{16}}{\cos^2\frac{1}{8}\theta\cos^2\frac{1}{16}\theta} + \&c.$$

FIRST SOLUTION. *By Prof. C. Avery, Hamilton College, Clinton.*

Here $\quad 1 + 4\cos^4\theta = 1 + (1 + \cos 2\theta)^2 = 2(1 + \cos 2\theta) + \cos^2 2\theta$
$\qquad = 4\cos^2\theta + \cos^2 2\theta,$
$1 + 4\cos^4\frac{1}{4}\theta = 4\cos^2\frac{1}{4}\theta + \cos^2\frac{1}{2}\theta,$
$1 + 4\cos^4\frac{1}{16}\theta = 4\cos^2\frac{1}{16}\theta + \cos^2\frac{1}{8}\theta,$
&c.;

and, by substitution, the given series is changed to

$$s_n = \frac{4}{\cos^2 2\theta} + \frac{1}{\cos^2\theta} + \frac{1}{4\cos^2\frac{1}{4}\theta} + \dots \frac{1}{4^{2n-2}\cos^2 4^{-n+1}\theta}$$

$$= \frac{d.}{d\theta}[2\tan 2\theta + \tan\theta + \tfrac{1}{4}\tan\tfrac{1}{4}\theta + \dots . 4^{-n+1}\tan 4^{-n+1}\theta]$$

$$= \frac{d.}{d\theta}[4^{-n+1}\cot 4^{-n+1}\theta - 4\cot 4\theta]$$

$$=\frac{16}{\sin^2 4\theta}-\frac{1}{4^{2n-2}\sin^2 4^{-n+1}\theta}.$$

And when $n=\infty$,

$$s=\frac{16}{\sin^2 4\theta}-\frac{1}{\theta^2}.$$

SECOND SOLUTION *By L., Murray Co., Geo.*

We have $u_{n+1}=4^{-2n}.\dfrac{1+4\cos^4 4^{-n}\theta}{\cos^2 4^{-n}\theta\cos^2 4^{-n}.2\theta}$

$$=4^{-2n}.\frac{4\cos^2 4^{-n}\theta+\cos^2 4^{-n}.2\theta}{\cos^2 4^{-n}\theta\cos^2 4^{-n}.2\theta}$$

$$=\frac{4^{-2n+1}}{\cos^2 4^{-n}.2\theta}+\frac{4^{-2n}}{\cos^2 4^{-n}\theta}$$

$$=\frac{4^{-2n+2}\sin^2 4^{-n}.2\theta}{\sin^2 4^{-n+1}\theta}+\frac{4^{-2n+1}\sin^2 4^{-n}\theta}{\sin^2 4^{-n}.2\theta}$$

$$=\frac{4^{-2n+2}(1-\cos^2 4^{-n}.2\theta)}{\sin^2 4^{-n+1}\theta}+\frac{4^{-2n+1}(1-\cos^2 4^{-n}\theta)}{\sin^2 4^{-n}.2\theta}$$

$$=\frac{4^{-2n+2}}{\sin^2 4^{-n+1}\theta}-\frac{4^{-2n+1}}{\sin^2 4^{-n}.2\theta}+\frac{4^{-2n+1}}{\sin^2 4^{-n}.2\theta}-\frac{4^{-2n}}{\sin^2 4^{-n}\theta}$$

$$=-\Delta.\frac{4^{-2n+2}}{\sin^2 4^{-n+1}\theta}$$

Hence $\quad s_n=\text{const.}+\Sigma u_{n+1}=\text{const.}-\dfrac{4^{-2n+2}}{\sin^2 4^{-n+1}\theta}$

$$s_0=0=\text{const.}-\frac{16}{\sin^2 4\theta};$$

$$s_n=\frac{16}{\sin^2 4\theta}-\frac{4^{-2n+2}}{\sin^2 4^{-n+1}\theta}.$$

$$s_\infty=\frac{16}{\sin^2 4\theta}-\frac{1}{\theta^2}.$$

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(107). QUESTION X. *By P.*

It is required to solve question (75) when, instead of the area and vertical angle, there are given the area and the side opposite the fixed extremity of the base.

SOLUTION. *By the Proposer.*

Retaining the notation in the solution of question (75); let BC $=a$, and the perpendicular from A upon BC $=h$; then

the equation of BC is $\quad y(x_2-x_,)=y_2(x-x_,)$,

that of the perp. from A upon CC, $\quad yy_2=(x_,-x_2)x$ . . . (1),

that of the perp. from C upon AB, $\quad x=x_2$ . . . . (2).

Moreover $\quad y_2^{\,2}+(x'-x_2)^2=a^2$ . . . . (3),

and $\quad y^2\,x_,=ah=2\Delta\,\text{ABC}$ . . . . . (4),

By eliminating x_1, x_2, y_2 among these four equations. we get

$$x^4(x^2+y^2) = h^2(x^2+y^2)^2 - 2ahxy(x^2+y^2) + a^2x^2y^2 \quad . \quad (5),$$

for the equation of the line, which is of the sixth order.

If we change these into polar co-ordinates, A being the pole, and the positive axis of x the angular axis, it becomes

$$v\cos^2\varphi = \pm(h - a\sin\varphi\cos\varphi) \quad . \quad . \quad . \quad (6).$$

The double sign indicates that the curve is the same in the opposite right angles, and we shall therefore use the upper one, and trace the curve from $\varphi = 0$ to $\varphi = \pi$; assume also k and β such that

$$k = \frac{a}{h}, \text{ and } \sin 2\beta = \frac{2}{k};$$

then

$$v = h\,.\,\frac{1 - \frac{1}{2}k\sin 2\varphi}{\cos^2\varphi},$$

$$x = h\,.\,\frac{1 - \frac{1}{2}k\sin 2\varphi}{\cos\varphi},$$

$$y = h\,.\,\frac{\sin\varphi - \frac{1}{2}k\sin\varphi\sin 2\varphi}{\cos^2\varphi},$$

$$\frac{dv}{d\varphi} = h\,.\,\frac{2\sin\varphi - k\cos\varphi}{\cos^3\varphi},$$

$$\frac{dx}{d\varphi} = h\,.\,\frac{\sin\varphi - k\cos^3\varphi}{\cos^2\varphi},$$

$$\frac{dy}{d\varphi} = h\,.\,\frac{1 + \sin^2\varphi - \frac{1}{2}k\sin 2\varphi(1 + \cos^2\varphi)}{\cos^3\varphi},$$

$$\frac{dy}{dx} = \frac{1 + \sin^2\varphi - \frac{1}{2}k\sin 2\varphi(1 + \cos^2\varphi)}{\cos\varphi(\sin\varphi - k\cos^3\varphi)}.$$

$$\frac{d^2y}{dx^2} = \frac{2 - 3\cos^2\varphi - 6k\sin\varphi\cos^3\varphi + k^2\cos^4\varphi(3 - \cos^2\varphi)}{h(\sin\varphi - k\cos^3\varphi)^3}.$$

The form of the curve will be seen by tracing it through the following points:—

1°. When $\varphi = 0°$, $v = x = h$, and if ψ be the angle made by the tangent at any point in the curve with the axis of x, for this point

$$\tan\psi = \frac{dy}{dx} = -\frac{1}{k}, \text{ and } \frac{d^2y}{dx^2} = \frac{1 - 2k^2}{hk},$$

As φ increases, v decreases, until

2°. When $k\sin 2\varphi = 2$, or $\varphi = \beta$, $v = x = y = 0$, $\tan\psi = \frac{dy}{dx} = \tan\beta$;

and the curve passes through the origin in the direction of the radius vector. As φ increases, v continues to decrease, until

3°. When $\frac{dv}{dx} = 0$, or $\tan\varphi = \frac{1}{2}k$; then

$$v = h(1 - \tfrac{1}{4}k^2), \tan\psi = \frac{dy}{dx} = -\frac{2}{k} = -\cot\varphi, \frac{d^2y}{dx^2} = \frac{(2k^4 + 20k^2 + 16)(k^2+4)^{\frac{3}{2}}}{hk^3(k^2-4)^2},$$

or the curve is perpendicular to the radius vector. Now v increases until

4°. When $k\sin 2\varphi = 2$ again, or $\varphi = \frac{1}{2}\pi - \beta$, then

$$v = x = y = 0, \tan\psi = \frac{dy}{dx} = \cot\beta,$$

and the curve again passes through the origin in the direction of the radius vector, and then v still continues to increase until

5°. When $\varphi = \frac{1}{2}\pi$, then $v = x = y = \infty$, $\tan\psi = \frac{dy}{dx} = \infty$, and there is no assymptote to this infinite branch. As φ increases from $\frac{1}{2}\pi$ to π, v decreases, until

6°. When $\varphi = \pi$, then

$$v = -x = h, \quad \tan\psi = \frac{dy}{dx} = -\frac{1}{k}, \quad \frac{d^2y}{dx^2} = \frac{2k^2-1}{hk},$$

7°. The curve is perpendicular to the axis of x, and x is a minimum when $\frac{dx}{d\varphi} = 0$, or, if $\tan\varphi = t$, when

$$t^3 + t - k = 0 \quad . \quad . \quad . \quad . \quad (7),$$

an equation which has never more than one real root.

8°. The curve is parallel to the axis of x, and y is a maximum or minimum, when $\frac{dy}{d\varphi} = 0$, or when

$$2t^4 - kt^3 + 3t^2 - 2kt + 1 = 0, \quad . \quad . \quad . \quad (8);$$

and examining this equation, by Sturm's Theorem, we find that if

$$x_1 = -32k^6 - 795k^4 + 2388k^2 + 32,$$

the equation has no real roots, when $x_1 > 0$, or $k^2 < 2{,}720732$,
two equal roots, when $x_1 = 0$, or $k^2 = 2{,}720732$,
two unequal roots, when $x_1 < 0$, or $k > 2{,}720732$;

and the roots are both positive, the corresponding values of φ being between 0 and $\frac{1}{2}\pi$.

9°. There are points of inflexion in the curve, when $\frac{d^2y}{dx^2} = 0$, or

$$2t^6 + 3t^4 - 6kt^3 + 3k^2t^2 - 6kt + 2k^2 - 1 = 0 \quad . \quad (9),$$

an equation which has no real roots, when $k > 2$,
two equal roots, when $k = 2$,
two unequal roots, when $k < 2$;

which two roots are both positive when $k > \frac{1}{2}$, one of them is zero, when $k = \frac{1}{2}$, they have different signs when $k < \frac{1}{2}$, and when $k = 0$, these roots $= \pm\sqrt{\frac{1}{2}}$.

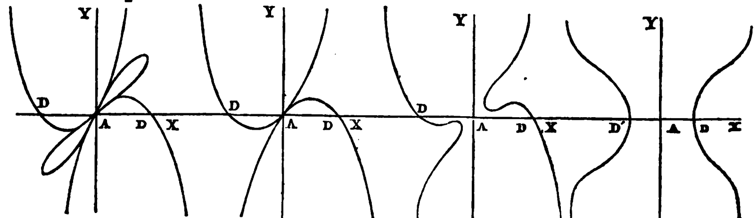

Fig. 1. Fig. 2. Fig. 3. Fig. 4.

Every relation between the constants a and h can be expressed by imagining $h = AD = AD'$ to remain constant, while their ratio, k, varies

from ∞ to 0. When $k = \infty$ the curve is confounded with the axes of co-ordinates; as k decreases, the curve takes the form of fig. 1, the loop continually decreasing in magnitude until $k = 2$, when it disappears, and the curve is as in fig. 2, when $k < 2$, the curve does not pass through the origin, and the curve will be as in fig. 3, approaching to the form of fig. 4, as k approaches to zero, which shape it finally assumes when $k = 0$, and the equation of this curve is

$$x^4 = h^2 (x^2 + y^2) \quad . \quad . \quad . \quad . \quad (10).$$

There are no points of inflection in fig. 1, the roots of equation (9), not being real when $k > 2$; the two real roots of equation (8) determine the position of the maximum ordinates in the loop and in the branch AD. In fig. 2, where $k = 2$; the points determined in 2°, 3°, 4°, 5°, 6°, 7°, one of of the points in 8°, and the two points in 9°, all coincide in the point A; the curve therefore does not, in its course, pass through this point, but there are two cusps whose vertices meet there. In fig. 3, the two points of inflection are on contrary sides of the point of least distance to A, and they recede from it as k decreases, one of them coinciding with the point D when $k^2 = \frac{1}{2}$, after which it is below the axis.

(108). QUESTION XI. *From Legendre's Theorie des Nombres, Vol. 2., p. 144.*
(Communicated by Mr. Geo. R. Perkins.)

A	B	C	D
E	F	G	H
I	K	L	M
N	O	P	Q

"In a square, divided into 16 spaces, as in the adjoining figure inscribe 16 numbers, A, B, C, ... Q, which will satisfy the following conditions:

1°. That the sum of the squares of the numbers may be equal in each of the four horizontal lines, also equal in each of the four vertical lines, and in the two diagonals.

2°. That the sum of the products, taken two and two, such as AE + BF + CG + DH may be equal to nothing with regard to the first two horizontal lines, as well as with regard to any two horizontal lines whatever, and that this may be the same also with regard to any two vertical lines."

First Solution. *By Prof. C. Avery.*

Let a, a', a'', a'''; b, b', b'', b'''; c, c', c'', c'''; d, d', d'', d'' be the required numbers, then

$$\left.\begin{aligned} A &= a'^2 + a'^2 + a''^2 + a'''^2 = b^2 + b'^2 + b''^2 + b'''^2 \\ &= c^2 + c'^2 + c''^2 + c'''^2 = d^2 + d'^2 + d''^2 + d'''^2 \end{aligned}\right\} \quad . \quad (1),$$

$$\left.\begin{aligned} A &= a^2 + b^2 + c^2 + d^2 = a'^2 + b'^2 + c'^2 + d'^2 \\ &= a''^2 + b''^2 + c''^2 + d''^2 = a'''^2 + b'''^2 + c'''^2 + d'''^2 \end{aligned}\right\} \quad . \quad (2),$$

$$A = a^2 + b'^2 + c''^2 + d'''^2 = a'''^2 + b''^2 + c'^2 + d^2 \quad . \quad . \quad (3),$$

$$\left.\begin{aligned} 0 &= ab + a'b' + a''b'' + a'''b''' = ac + a'c' + a''c'' + a'''c''' \\ &= ad + a'd' + a''d'' + a'''d''' = bc + b'c' + b''c'' + b'''c''' \\ &= bd + b'd' + b''d'' + b'''d''' = cd + c'd' + c''d'' + c'''d''' \end{aligned}\right\} \quad . \quad . \quad . \quad (4),$$

$$\left.\begin{aligned} 0 &= aa' + bb' + cc' + dd' = aa'' + bb'' + cc'' + dd'' \\ &= aa''' + bb''' + cc''' + dd''' = a'a'' + b'b'' + c'c'' + d'd'' \\ &= a'a''' + b'b''' + c'c''' + d'd''' = a''a''' + b''b''' + c''c''' + d''d''' \end{aligned}\right\} \quad . \quad (5),$$

If (1) and (4) be satisfied, (2) and (4) will be :—See La Place, page 115. In order to satisfy (1) and (4) we use the well known principles demonstrated by Barlow, in his Theory of Numbers, (page 179); to wit:

$$(p^2+q^2+r^2+s^2)(p'^2+q'^2+r'^2+s'^2)=(pp'+qq'+rr'+ss')^2$$
$$+(pq'-qp'+rs'-sr')^2+(pr'-rp'+sq'-qs')^2+(ps'-sp'+qr'-rq')^2,$$

and where the signs of the simple quantities may be changed at pleasure. Assume

$$\left.\begin{array}{ll} a = pr'+qs'+rp'+sq', & a' = -pp'+qq'+rr'-ss' \\ a'' = ps'-qr'+rq'-sp', & a''' = -pq'-qp'+rs'+sr' \end{array}\right\} (6),$$
$$\left.\begin{array}{ll} b = -pq'+qp'-rs'+sr', & b' = ps'+qr'-rp'-sp' \\ b'' = pp'+qq'+rr'+ss', & b''' = -pr'+qs'+rp'-sq' \end{array}\right\} (7),$$
$$\left.\begin{array}{ll} c = pp'+qq'-rr'-ss', & c' = pr'-qs'+rp'-sq' \\ c'' = pq'-qp'-rs'+sr', & c''' = ps'+qr'+rq'+sp' \end{array}\right\} (8),$$
$$\left.\begin{array}{ll} d = -ps'+qr'+rq'-sp', & d' = -pq'-qp'-rs'-sr' \\ d'' = pr'+qs'-rp'-sq', & d''' = pp'-qq'+rr'-ss' \end{array}\right\} (9).$$

Then will all the equations be satisfied, save (3). From (1) and (3)

$$0=b'^2+c''^2+d'''^2-a'^2-a''^2-a'''^2=b''^2+c'^2+d^2-a^2-a'^2-a''^2 \quad (10).$$

If we write the previous values of a, a', &c. in these two equations, and take their sum and difference, putting $s=0$, we shall find

$$p'r'=q's', \quad \text{and} \quad p\,(p'q'-r's')=r\,(p's'-q'r') \quad . \quad (11),$$

$$\text{Whence } p'=\frac{q's'}{r'}, \quad \text{and } p=\frac{rq'}{s'}\cdot\frac{s'^2-r'^2}{q'^2-r'^2} \qquad (12),$$

and $q'r's'$, &c. may be taken at pleasure. For instance, ifg $r=5$, $s=0$, $q'=3$, $r'=2$, $s'=4$; then $p=9$, $p'=6$, and the numbers placed in their appropriate cells will be

$48+4q$	$-44+3q$	$51-2q$	$-7-6q$
$-47+6q$	$21+2q$	$64+3q$	$12+4q$
$44+3q$	$48-4q$	$7-6q$	$51+2q$
$-21+2q$	$-47+6q$	$-12+4q$	$64-3q$

If $q=0$, 1, 2, &c., we have different Tables which will satisfy. When $q=5$ we obtain Euler's numbers

68	—29	41	—37
—17	31	79	32
59	28	—23	61
—11	—77	8	49

Second Solution. *By Prof. G. R. Perkins, Utica Academy.*

a	a'	a''	a'''
b	b'	b''	b'''
c	c'	c''	c'''
d	d'	d''	d'''

For greater symmetry, we will represent the numbers as in the adjoining figure.

Now by the conditions of the question, we must satisfy the following equations, where A denotes any constant quantity:

$$\left.\begin{array}{llllll} a^2 +a'^2 +a''^2 +a'''^2 = \mathrm{A}, & b^2 +b'^2 +b''^2 +b'''^2 = \mathrm{A}, \\ c^2 +c'^2 +c''^2 +c'''^2 = \mathrm{A}, & d^2 +d'^2 +d''^2 +d'''^2 = \mathrm{A}, \end{array}\right\} \quad (1)$$

$$\left.\begin{array}{ll} a^2 +b^2 +c^2 +d^2 = \mathrm{A}, & a'^2 +b'^2 +c'^2 +d'^2 = \mathrm{A}, \\ a''^2 +b''^2 +c''^2 +d''^2 = \mathrm{A}, & a'''^2 +b'''^2 +c'''^2 +d'''^2 = \mathrm{A}, \end{array}\right\} \quad (2),$$

$$\left.\begin{array}{ll} ab +a'b' +a''b''+a'''b''' =0, & ac +a'c' +a''c'' +a'c''' =0, \\ ad +a'd' +a''d''+a'''d''' =0, & bc +b'c' +b''c'' +b'''c''' =0, \\ bd +b'd' +b''d''+b'''d''' =0, & cd +c'd' +c''d'' +c'''d''' =0, \end{array}\right\} \quad (3),$$

$$\left.\begin{array}{ll} aa' +bb' +ec' +dd' =0, & aa'' +bb'' +cc'' +dd'' =0, \\ aa''' +bb''' +cc''' +dd''' =0, & a'a'' +b'b'' +c'c'' +d'd'' =0, \\ a'a'''+b'b'''+c'c''' +d'd''' =0. & a''a'''+b''b'''+c''c'''+d''d'''=0, \end{array}\right\} \quad (4),$$

$$a^2 +b'^2 +c''^2 +d'''^2 = \mathrm{A}, \qquad a'''^2 +b''^2 +c'^2 +d^2 = \mathrm{A}, \quad (5),$$

We will first show that if equations (1), (3) be satisfied, then will (2) and (4) be also satisfied: for assume

$$\left.\begin{array}{ll} w = as +bt +cu +dv, & x=a's +b't +c'u +d'v \\ y = a''s +b''t+c''u+d''v, & z=a'''s+b'''t+c'''u+d'''v \end{array}\right\} \quad (6).$$

If we take the sum of the squares of these four equations, we shall have in virtue of (4) and (3)

$$w^2+x^2+y^2+z^2 = \mathrm{A}\,(s^2+t^2+w^2+v^2) \quad . \quad . \quad . \quad (7).$$

From (6) we get, by having to (1) and (3),

$$\left.\begin{array}{ll} \mathrm{A}\,s =aw+a'x+a''y+a'''z, & \mathrm{A}\,t=bw+b'x+b''y+b'''z, \\ \mathrm{A}\,u=cw +c'x +c''y +c'''z, & \mathrm{A}\,v=dw+d'x+d''y+d'''z, \end{array}\right\} \quad (8).$$

If we substitute these values in (7), and compare the like co-efficients, we shall obtain (2) and (4). Hence we need only seek to satisfy (1), (3) and (5).

On page 213, Vol. 1., of *Theorie des Nombres*, we find this identical equation given by Euler,

$$(p^2+q^2+r^2+s^2)(p'^2+q'^2+r'^2+s'^2) = (pp'+qq'+rr'+ss')^2$$
$$+(pr'+qs'-rp'-sq')^2+(ps-qr'+rq'-sp')^2+(pq'-qp'-rs'+sr')^2 \, (9)$$

This equation will still hold good, when the signs of any of the letters are changed. The four terms on the right hand member of (9) may have their sign so changed as to correspond with the values of a, a', a'', a''', and of b, b', b'', b''', and of c, c', c'', c''', and of d, d', d'', d''' in succession. This is done in the following manner: Form a square of 16 cells, and write the four terms in the same order which they have in (9) for the first horizontal line; then for the second horizontal line, write these same terms in a reverse order, observing to change the signs of r, s in the first two terms, and the signs of p, q in the last two terms. The remaining cells are filled by taking the terms in the first horizontal line for the third line, those in the second line for the fourth, placing each term, either one vertical column to the right, or one to the left, but not crossing the middle vertical line, observing to change the signs of q, s, when the term is moved to the right, and to change the signs of p, r when it is moved to the left.

$pp'+qq'+rr'+ss'$	$pr'+qs'-rp'-sq'$	$ps'-qr'+rq'-sp'$	$pq'-qp'-rs'+sr'$
$-pq'+qp'-rs'+sr'$	$-ps'+qr'+rq'-sp'$	$pr'+qs'+rp'+sq'$	$pp'+qq'-rr'-ss'$
$-pr'+qs'+rp'-sq'$	$pp'-qq'+rr'-ss'$	$-pq'-qp'+rs'+sr'$	$ps'+qr'+rq'+sp'$
$ps'+qr'-rq'-sp'$	$-pq'-qp'-rs'-sr'$	$-pp'+qq'+rr'-ss'$	$pr'-qs'+rp'-sq'$

$pr'+qs'+rp'+sq'$	$-pp'+qq'+rr'-ss'$	$ps'-qr'+rq'-sp'$	$-pq'-qp'+rs'+sr'$
$-pq'+qp'-rs'+sr'$	$ps'+qr'-rq'-sp'$	$pp'+qq'+rr'+ss'$	$-pr'+qs'+rp'-sq'$
$pp'+qq'-rr'-ss'$	$pr'-qs'+rp'-sq'$	$pq'-qp'-rs'+sr'$	$ps'+qr'+rq'+sp'$
$-ps'+qr'+rq'-sp'$	$-pq'-qp'-rs'-sr'$	$pr'+qs'-rp'-sq'$	$pp'-qq'+rr'-ss'$

The terms may be permuted in a great variety of ways, by observing the above law of filling the cells; either of the above squares will satisfy the conditions of the question, except (5).

It remains to satisfy equations (5); for this purpose we will subtract the first of equations (1) from equations (5), and we get

$$b'^2+c''^2+d'''^2-a'^2-a''^2-a'''^2=0,\ b''^2+c'^2+d^2-a^2-a'^2-a''^2=0 \quad . \quad . \quad . \quad . \quad . \quad (10).$$

Substituting for these values their corresponding values in the last square, and taking the sum and difference of the resulting equations, we get

$$\left.\begin{aligned} pq(p'q'-r's')+ps(p's'-q'r')+qr(q'r'-p's')+rs(r's'-p'q)=0 \\ pr(p'r'-q's')+qs(q's'-p'r')=0 \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11).$$

If we make $s=0$, these become

$$\left.\begin{aligned} p'r'=q's' \\ p(p'q'-r's')=r(p's'-q'r') \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12).$$

68	—29	41	—37
—17	31	70	32
59	28	—23	61
—11	—77	8	49

Equations (12) may be satisfied in an infinite number of ways. If we take

$$p=9,\ q=5,\ r=5,\ s=0,$$
$$p'=6,\ q'=3,\ r'=2,\ s'=4;$$

we shall get the adjoining square, which are smaller than the numbers given by Euler.

$42+2q$	$-11+4q$	$24-q$	$2-8q$
$-18+8q$	$-16+q$	$21+4q$	$38+2q$
$11+4q$	$42-2q$	$-2-8q$	$24+q$
$16+q$	$-18-8q$	$-38+2q$	$21-4q$

If $p=2$, $q=q$, $r=5$, $s=0$, $p'=8$, $q'=4$, $r'=1$, $s'=2$; the sixteen numbers will be as in the margin, and in which q is entirely arbitrary. If $q=3$, we get the first of the succeeding sets; and if $p=9$, $q=2$, $r=5$, $s=0$, $p'=6$, $q'=3$, $r'=2$, $s'=4$, we have the second of the sets, both less than the numbers of Euler:

48	1	21	—22
6	—13	33	44
23	36	—26	27
19	—42	—32	9

56	—38	47	—19
—35	25	70	20
50	40	— 5	55
—17	—59	— 4	58

It should be observed, that these numbers, besides fulfilling the conditions of the question, satisfy the following:

1°. The sum of the squares of the four central terms, as well as the sum of the squares of the four corner terms, is equal to the sum of the squares of any four terms in a row.

2°. The sum of the squares of the four terms, at the extremities of the two middle horizontal rows, as well as the sum of the squares of the four terms, at the extremities of the two middle vertical rows, is the same as the sum of the squares of any four terms in a row.

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(109). Question XII. *By* ——.

It is required to find the locus of the centres, and the *envelope*, of all the spheres that can be made to touch the surface of a given sphere, and also two planes, given in position.

Solution. *By the Proposer.*

The centres of the spheres will be in a plane bisecting the angle 2θ, of the given planes; let this plane be the plane of xy, and let the plane of xz pass through the centre of the sphere, the axis of y being the intersection of the planes. Let the centre of the given sphere be $a, 0, c$ and its radius R; the centre of one of the tangent spheres $x, y, 0$ and its radius r. Then, since the sphere is tangent to the planes,

(1), $\quad r = x \sin\theta,$

and since they are tangent to each other

(2) $\quad (x-a)^2+y^2+c^2=(R+r)^2;$

and eliminating r between these equations, we get the equation of the locus of the centres of the touching spheres

(3), $\quad y^2+x^2\cos^2\theta-2kx-A^2=0;$

which is therefore an ellipse, having its major axis on the axis of x, the distance of its centre from the origin being $k\sec^2\theta$, and the length of its semiaxes

$M\sec\theta$ and $M\sec^2\theta$, where

(4), $k=a+R\sin\theta, A^2=R^2-a^2-c^2, M^2=k^2+A^2\cos^2\theta=(R+a\sin\theta)^2-c^2\cos^2\theta$

If the given sphere is either enveloped by, or envelopes, the tangent

spheres, R must be taken negative in these equations. If the given sphere intersects either or both the planes, there will be tangent spheres having their centres in the plane of yz, and the locus of their centres will be had by writing z for x and $\frac{1}{2}\pi - \theta$ for θ in equation (3).

The equation of the touching sphere whose centre is yx, is

(5). $$(X - x)^2 + (Y - y)^2 + Z^2 = r^2 = x^2 \sin^2\theta,$$

and that of the next one whose centre is $(y+dy),(x+dx)$, is

$$(X - x - dx)^2 + (Y - y - dy)^2 + Z^2 = (x+dx)^2 \sin^2\theta$$

and therefore, for the intersection of these two spheres,

$$(Y - y)\, dy + (X - x \cos^2\theta)\, dx = 0;$$

but by (3), $$y\, dy - (k - x\cos^2\theta)\, dx = 0,$$

and by eliminating dy and dx.

(6), $$k\,(Y - y) - Yx\cos^2\theta + Xy = 0.$$

(5) and (6) are the equations of the intersection of two consecutive tangent spheres, and eliminating y and x between (3), (5), (6), we get the equation of the envelope of all the tangent spheres, and which can be put into either of the two forms

(7), $$\{(X^2+Y^2+Z^2+A^2)\cos^2\theta + 2k(k - X)\}^2 - 4M^2\{(k - X)^2 + Y^2\cos^2\theta\} = 0,$$

(8), $$\{(X^2+Y^2+Z^2-A^2)\cos^2\theta - 2kX\}^2 - 4M^2\{X^2\sin^2\theta - Z^2\cos^2\theta\} = 0$$

When $X = 0$, equation (8) becomes

$$(Y^2+Z^2-A^2)^2\cos^4\theta + 4M^2Z^2\cos^2\theta = 0,$$

which can only be satisfied by the values

$$Z = 0,\ Y = \pm A = \pm\sqrt{R^2 - a^2 - c^2},$$

and these two points are the intersection of the surface with the plane YZ; they only exist when

$$R^2 = \text{or} > a^2 + c^2,$$

or when the axis of Y is either a tangent or chord of the given sphere.

When $Z = 0$, equation (8) represents the intersection of the surface with the plane of XY; its first member is divisible into two factors, and the equation is satisfied by either

(9), $$Y^2\cos^4\theta + (X\cos^2\theta - k - M\sin\theta)^2 = (M + k\sin\theta)^2,$$

(10), $$Y^2\cos^4\theta + (X\cos^2\theta - k + M\sin\theta)^2 = (M - k\sin\theta)^2;$$

which are two circles such that, R_1, R_2 being their radii, and D the distance of their centres,

(11), $$R_1 = \frac{M + k\sin\theta}{\cos^2\theta},\ R_2 = \pm\frac{M - k\sin\theta}{\cos^2\theta},\ D = \frac{2M\sin\theta}{\cos^2\theta}.$$

The upper sign is used when

$$M > k\sin\theta,$$
$$k^2 + A^2\cos^2\theta > k^2\sin^2\theta,$$
$$k^2 + A^2 > 0,$$
$$R^2(1 + \sin^2\theta) + 2aR\sin\theta - c^2 > 0;$$

and in this case

$$R_1 + R_2 = \frac{2M}{\cos^2\theta} > d,$$

$$R_1 - R_2 = \frac{2k\sin\theta}{\cos^2\theta} = \frac{kD}{M};$$

hence the circles intersect each other when

$$k < M, \text{ or } R^2 > a^2 + c^2,$$

they touch each other when

$$k = \text{M}, \text{ or } \text{R}^2 = a^2 + c^2,$$

and they envelope each other when

$$k > \text{M}, \text{ or } \text{R}^2 < a^2 + c^2.$$

The under sign is used when

$$\text{M} < k \sin\theta,$$

$$\text{R}^2 (1 + \sin^2\theta) + 2a\text{R}\sin\theta - c^2 < 0;$$

and then $$\text{R}_1 - \text{R}_2 = \frac{2\text{M}}{\cos^2\theta} > \text{D},$$

or the circles envelope each other. One circle becomes a point when

$$\text{M} = k\sin\theta,$$

or $$\text{R}^2 (1 + \sin^2\theta) + 2a\text{R}\sin\theta - c^2 = 0.$$

If, then, it were required under any circumstances, to make spheres touch a given sphere and two planes; it would be sufficient to make circles touch the two circles (9) and (10), under the same circumstances, and the spheres of which these touching circles are the great sections, would evidently be the ones required. For instance, if spheres were described on the tangent circles in question (50), they would all touch a sphere and two planes whose positions may be found. It follows also that, having given two planes and a sphere, the sphere may be so placed that n spheres can be described each touching two of the others as well as the given sphere and the two planes; it is obviously sufficient that R_1, R_2, D may have the relation of R, r, d in equation (25), p. 248, Vol. I.; and this will be the case when the centre of the sphere is placed on the circumference of either the ellipse or the hyperbola, on the plane of xz, whose equations are

$$c^2\left(1 - \cos^2\theta\cos^2\frac{i\pi}{n}\right) + a^2\sin^2\theta\cos^2\frac{i\pi}{n} - 2a\text{R}\sin\theta\sin^2\frac{i\pi}{n}$$
$$= \text{R}^2\left(\sin^2\theta + \sin^2\frac{i\pi}{n}\right),$$

$$c^2\left(1 - \sin^2\theta\cos^2\frac{i\pi}{n}\right) - a^2\sin^2\theta\cos^2\frac{i\pi}{n} - 2a\text{R}\sin\theta$$
$$= \text{R}^2\left(1 + \sin^2\theta\sin^2\frac{i\pi}{n}\right).$$

Again, when $\text{Y} = 0$, the equation (7) represents the intersection of the surface with the plane of XY; and it is satisfied by either

(12), $$\text{Z}^2\cos^4\theta + (\text{X}\cos^2\theta - \text{M} - k)^2 = (\text{M} + k)^2\sin^2\theta,$$

(13), $$\text{Z}^2\cos^4\theta + (\text{X}\cos^2\theta + \text{M} - k)^2 = (\text{M} - k)^2\sin^2\theta;$$

which are also two circles, and if r_1, r_2 be their radii, d the distance of their centres we shall have

(14), $$r_1 = \frac{(\text{M}+k)\sin\theta}{\cos^2\theta}, \quad r_2 = \frac{(\text{M}-k)\sin\theta}{\cos^2\theta}, \quad d = \frac{2\text{M}}{\cos^2\theta};$$

and we find that

when $\text{M} > k\sin\theta$, these circles are without each other;

when $\text{M} = k\sin\theta$, they touch externally;

when $\text{M} < k\sin\theta$, they intersect each other.

The surface is evidently an elliptic ring, its axes being the ellipse (3)

on the plane of XY, and all the sections perpendicular to this ellipse are circles, whose equations are (5) and (6).

(110.) QUESTION XIII. *By* ψ.

See Dr. Bowditch's Commentary on the Mécanique Céleste, Vol. I, page 304, equations (i). It is required to be determined whether these equations cannot be reduced to the forms given in page 313, equations (H), in a more simple manner than has been done in that admirable work.

FIRST SOLUTION. *By Dr. T. Strong, New-Brunswick, N. J.*

The equations (i) are

$$\frac{d^2x}{dt^2}=\frac{dQ}{dx},\qquad \frac{d^2y}{dt^2}=\frac{dQ}{dy},\qquad \frac{d^2z}{dt^2}=\frac{dQ}{dz}\quad . \qquad (1),$$

Q being a function of x, y, z; and at p. 306 of that work,

$$x=r\cos\theta\cos v,\qquad y=\text{R}\cos\theta\sin v,\qquad z=r\sin\theta\quad . \qquad (2);$$

hence Q is a function of r, θ, v; therefore

$$\frac{dQ}{dx}dx+\frac{dQ}{dy}dy+\frac{dQ}{dz}dz=\frac{dQ}{dr}dr+\frac{dQ}{d\theta}d\theta+\frac{dQ}{dv}dv\quad . \quad (3);$$

or, by substituting the values of dx, dy, dz from (2) in (3), we get

$$-\left[\frac{dq}{dx}r\sin\theta\cos v+\frac{dQ}{dy}r\sin\theta\sin v-\frac{dQ}{dz}r\cos\theta\right]d\theta$$

$$+\left[\frac{dQ}{dy}r\cos\theta\cos v-\frac{dQ}{dx}r\cos\theta\sin v\right]+\left[\frac{x}{r}.\frac{dQ}{dx}+\frac{y}{r}.\frac{dQ}{dy}+\frac{z}{r}.\frac{dQ}{dz}\right]dr$$

$$=\frac{dQ}{d\theta}d\theta+\frac{dQ}{dv}dv+\frac{dQ}{dr}dr;$$

therefore by the method of indeterminate co-efficients, equating the coefficients of $d\theta$, dv, dr, we have

$$\left.\begin{array}{l}
\dfrac{dQ}{dz}r\cos\theta-\dfrac{dQ}{dx}r\sin\theta\cos v-\dfrac{dQ}{dy}r\sin\theta\sin v=\dfrac{dQ}{d\theta},\\
\dfrac{dQ}{dy}r\cos\theta\cos v-\dfrac{dQ}{dx}r\cos\theta\sin v=x\dfrac{dQ}{dy}-y\dfrac{dQ}{dx}=\dfrac{dQ}{dv},\\
\text{or, by (1),}\quad \dfrac{xd^2y-yd^2x}{dt^2}=\dfrac{d(xdy-ydx)}{dt^2}=\dfrac{dQ}{dv},\\
x\dfrac{dQ}{dx}+y\dfrac{dQ}{dy}+z\dfrac{dQ}{dz}=r\dfrac{dQ}{dr}
\end{array}\right\}\quad (4).$$

By (1), the last of (4) becomes

$$\frac{xd^2x+yd^2y+zd^2z}{dt^2}=r\frac{dQ}{dr}\quad . \quad . \quad . \quad . \quad (5),$$

and, by (2), $x^2+y^2+z^2=r^2$, therefore $xdx+ydy+zdz=rdr$;
and $xd^2x+yd^2y+zd^2z=rd^2r+dr^2-(dx^2+dy^2+dz^2)$,
or, since, by (2), $dx^2+dy^2+dz^2=dr^2+r^2\cos^2\theta\, dv^2+r^2d\theta^2$,

$$xd^2x+yd^2y+zd^2z=rd^2r-r^2(\cos^2\theta\, dv^2+d\theta^2);$$

hence (5) is changed to

$$\frac{d^2r-r\cos^2\theta\, dv^2-rd\theta^2}{dt^2}=\frac{dQ}{dr}\quad . \quad . \quad (6),$$

which is the first of the equations (H). Again by (2),

$\frac{y}{x} = \tan v$, $\therefore \frac{xdy - ydx}{x^2} = \frac{dv}{\cos^2 v}$, or $xdy - ydx = r^2 \cos^2\theta\, dv$,

hence the last but one of equations (4) becomes

$$\frac{d\,(r^2\cos^2\theta dv)}{dt^2} = \frac{dQ}{dv} \quad . \quad . \quad . \quad . \quad (7),$$

which is the second of equations (H). The first of (4) is reduced by (1) to

$$\frac{r\cos\theta\, d^2 z - r\sin\theta \cos v d^2 x - r\sin\theta\ \sin v d^2 y}{dt^2} = \frac{dQ}{d\theta}$$

the first member of which is reduced by (2), to

$$\frac{r\cos^2\theta\, d^2 z - \sin\theta\,(xd^2x + yd^2y)}{dt^2\cos\theta} = \frac{rd^2z - \sin\theta\,(xd^2x + yd^2y + zd^2z)}{dt^2\cos\theta}$$

$$= \frac{rd^2z - \sin\theta\,[rd^2r - r^2\cos\theta^2 dv^2 - r^2 d\theta^2]}{dt^2\cos\theta}$$

$$= \frac{r^2 d^2\theta + 2rdr\, d\theta + r^2\sin\theta\cos\theta\, dv^2}{dt^2}$$

hence $$\frac{d\,(r^2 d\theta) + r^2\sin\theta\ \cos\theta\ dv^2}{dt^2} = \frac{dQ}{d\theta} \quad . \quad . \quad . \quad (8),$$

which is the third and last of equations (H). In conclusion we would remark that these equations could be obtained with the greatest facility by the very ingenious method of transformation given by Pontecoulant at p. 207, Vol. I, of his Systeme du Monde, which comes to the same thing as the method given by La Grange at p. 304, etc., of his Mec. Anal. Vol. I.

Second Solution. *By Prof. M. Catlin, Hamilton College.*

Put $p = \cos\theta, p' = \sin\theta,\ q = \cos v,\ q' = \sin v$. Then equations (*i*) are reduced, by the method of Dr. Bowditch, to the following:

$$\left.\begin{array}{ll} [503a] & \left(\frac{dQ}{dr}\right) = pq\,\frac{d^2x}{dt^2} + pq'\,\frac{d^2y}{dt^2} + p'\,\frac{d^2z}{dt^2} \\ [506a] & \left(\frac{dQ}{dv}\right) = -rpq'\,\frac{d^2x}{dt^2} + rpq\,\frac{d^2q}{dt^2} \\ [507a] & \left(\frac{dQ}{d\theta}\right) = -rp'\,\frac{d^2x}{dt^2} - rp'q'\,\frac{d^2y}{dt^2} + rp\,\frac{d^2z}{dt^2} \end{array}\right\} \quad (1),$$

and, (501), $\quad x = rpq, \quad y = rpq', \quad z = rp' \quad . \quad . \quad (2),$

$$\because \left.\begin{array}{l} d^2x = pqd^2r + prd^2q + qrd^2p + 2pdqdr + 2qdpdr + 2rdpdq \\ d^2y = pq'd^2r + prd^2q' + q'rd^2p + 2pdq'dr + 2q'dpdr + 2rdpdq' \\ d^2z = p'd^2r + rd^2p' + 2dp'dr \end{array}\right\} \quad (3).$$

Substitute (3) in (1), observing that $p^2 + p'^2 = 1, q^2 + q'^2 = 1$,

$$\left.\begin{array}{l} \left(\frac{dQ}{dr}\right) = \frac{d^2r}{dt^2} - rp^2 . \frac{dv^2}{dt^2} - r\frac{d\theta^2}{dt^2} \\ \left(\frac{dQ}{dv}\right) = \frac{d(r^2p^2dv)}{dt^2} \\ \left(\frac{dQ}{d\theta}\right) = r^2\,\frac{d^2\theta}{dt^2} + r^2\,pp' . \frac{dv^2}{dt^2} + 2r\frac{dr}{dt} . \frac{d\theta}{dt} \end{array}\right\} \quad . \quad . \quad (4);$$

which are the equations (H) required.

(111.) QUESTION XIV. *By Professor B. Peirce, Harvard University.*

Calling the evolute of a curve its first evolute, the evolute of the first evolute the second evolute, that of the second evolute the third evolute, and that of the third evolute the fourth evolute; to find a curve whose fourth evolute is the curve placed in a position parallel to its original one: *i. e.* one in which the equation is the same when referred to rectangular axes parallel in the one case to those in the other.

SOLUTION. *By the Proposer.*

Let $s =$ arc of given curve,
$\rho =$ its radius of curvature,
$\varphi =$ the angle which ρ makes with the axis;
and let s', ρ', φ' be the corresponding quantities for the fourth evolute. We find as in question (94),

$$\varphi' = \varphi + n \,.\, 360^\circ., \qquad \rho' = \frac{d^4\rho}{d\varphi^4},$$

in which n is any integer whatever; so that if the curve is determined by the equation $\rho = \mathrm{f}(\varphi)$,
we have

$$\mathrm{f}(\varphi + n \,.\, 360^\circ) = \frac{d^4 . \mathrm{f}(\varphi)}{d\varphi^4}.$$

If, now, we suppose

$$\rho = \mathrm{f}(\varphi) = \mathrm{A}e^{m\phi} + \mathrm{A}'e^{m'\phi} + \&c.,$$

we have A, A', &c., arbitrary, and m, m', &c., are the different roots of the equation $e^{mn.360^\circ} = m^4$.

General Case.—Let $n = 0$,
we have $m = \pm 1$, and $\pm \sqrt{-1}$,

$$\rho = \mathrm{A}e^{\phi} + \mathrm{A}'e^{-\phi} + \mathrm{B}\sin(\varphi + \beta),$$

in which A, A', B, β are arbitrary.

Case 1. Let $\mathrm{A}' = \mathrm{B} = 0$, the curve is a logarithmic spiral.

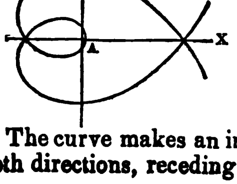
Case 2. Let $\mathrm{A} = \mathrm{A}' = 0$, the curve is a cycloid.

Case 3. Let $\mathrm{B} = 0$, the curve is a variety of Example 3, quest. (94).

Case 4. Let $\mathrm{A} = \mathrm{A}' = -\frac{1}{2}\mathrm{B}$, $\beta = \frac{1}{2}\pi$, then

$$\rho = \mathrm{A}\,(e^{\phi} + e^{-\phi} - 2\cos\varphi),$$

$$\begin{aligned} y &= \textstyle\int \rho d\varphi . \cos\varphi \\ &= \mathrm{A}\sqrt{\tfrac{1}{2}}\{e^{\phi}\sin(\varphi + 45^\circ) + e^{-\phi}\cos(\varphi - 45^\circ)\} \\ &\qquad - \mathrm{A}\,(\varphi + \tfrac{1}{2}\sin 2\varphi) \\ x &= -\textstyle\int \rho d\varphi . \sin\varphi \\ &= \mathrm{A}\sqrt{\tfrac{1}{2}}\{e^{\phi}\cos(\varphi + 45^\circ) + e^{-\phi}\cos(\varphi - 45^\circ)\} - \mathrm{A}\cos^2\varphi. \end{aligned}$$



When $\varphi = 0$, $\rho = y = x = 0$, $\frac{dy}{dx} = \infty$, the axis of y is a tangent to the curve; but we shall find also for two numerically equal values of φ, the one positive, the other negative,

$$y_\phi = -y_{-\phi}, \quad x_\phi = x_{-\phi}, \quad \rho_\phi = \rho_{-\phi};$$

and therefore there is no cusp at the origin A. The curve makes an infinite number of revolutions round this point in both directions, receding very rapidly from it.

(112). QUESTION XV. *By Prof. B. Peirce.*

Integrate the equations

$$\frac{d^2y}{dx^2}+\frac{A}{x}\cdot\frac{dy}{dx}-B^2\,x^ny=0,$$

$$\frac{d^2y}{dx^2}+A.\ \frac{dy}{dx}-B^2\,e^{nx}y=0;$$

in which A, B and n are constants, and e is the base of the Naperian system of logarithms.

FIRST SOLUTION. *By the Proposer.*

I. Since these equations are linear with respect to y, $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$, we have only to find two particular values to satisfy them. If we denote these values by y' and y'', and take C and C$'$ for two arbitrary constants, we have, for the complete value of y,

$$y=\text{C}\,y'+\text{C}'\,y''.$$

To find y' and y'' we will take the general equation of the second order

$$\frac{d^2y}{dx^2}+\text{P}\,\frac{dy}{dx}+\text{Q}\,y=0,$$

in which P and Q are functions of x. Suppose, then,

$$y'=\int e^{-\theta z}\varphi d\theta,$$

where z is a function of x, φ of θ. and the integral is definite. Then

$$\frac{dy'}{dx}=-\int\frac{dz}{dx}\,e^{-\theta z}\varphi\theta d\theta,$$

$$\frac{d^2y'}{dx^2}=-\int\frac{d^2z}{dx^2}\,e^{-\theta z}\,\varphi\theta d\theta+\int\frac{dz^2}{dx^2}\,e^{-\theta z}\,\varphi\theta^2\,d\theta$$

$$0=\frac{d^2y'}{dx^2}+\text{P}\,\frac{dy'}{dx}+\text{Q}y'$$

$$=\frac{dz^2}{dx^2}\int e^{-\theta z}\,\varphi\theta^2\,d\theta-\left(\frac{d^2z}{dx^2}+\text{P}\,\frac{dz}{dx}\right)\int e^{-\theta z}\,\varphi\theta\,d\theta+\text{Q}\int e^{-\theta z}\,\varphi\,d\theta.$$

We will suppose the second member of this last equation

$$=\text{X}\,.\ \Phi\,.\,e^{-\theta z},$$

in which X is a function x, and Φ of θ. Differentiation, relative to θ gives

$$\frac{dz^2}{dx^2}\,\varphi\theta^2-\left(\frac{d^2z}{dx^2}+\text{P}\,\frac{dz}{dx}\right)\varphi\,\theta+\text{Q}=\text{X}\,\left(\frac{d\Phi}{d\theta}-\Phi\,z\,\right).$$

If we again suppose $\Phi=a\varphi+b\varphi\theta+c\varphi\theta^2,$

$$\frac{d\Phi}{d\theta}=a'\varphi+b'\varphi\theta+c'\varphi\theta^2;$$

in which a, b, c, a', b', c' are constants, φ is determined by the equation

$$\frac{d\varphi}{\varphi}=\frac{a'-b+(b'-2c)\,\theta+c'\theta^2}{a+b\theta+c\theta^2}.\,d\theta\quad .\ .\ .\ .\ .\ \text{(I)};$$

and if we put the co-efficients of θ and θ^2 equal to zero, we have

$$\frac{dz^2}{dx^2} = X(c' - cz),$$

$$\frac{d^2z}{dx^2} + P\frac{dz}{dx} = -X(b' - bz),$$

$$Q = X(a' - az);$$

and, by the elimination of X,

$$\frac{dz^2}{dx^2}(a' - az) = Q(c' - cz) \quad . \quad . \quad . \quad . \quad . \quad \text{(II)},$$

$$\frac{d^2z}{dx^2} + P\frac{dz}{dx} = -\frac{dz^2}{dx^2}\,\frac{b' - bz}{c' - cz} \quad . \quad . \quad . \quad . \quad \text{(III)}.$$

Equations (II) and (III) serve to determine the value of z, when P and Q are given so as to satisfy the equation of condition involved in these equations, or to find values of P and Q for given values of z.

Case 1. Suppose, in (II) and (III), $\quad z = x^n$,

and they give

$$Q = n^2 x^{2n-2}.\frac{a' - ax^n}{c' - cx^n},$$

$$P = -nx^{n-1}.\frac{b' - bx^n}{c' - cx^n} - (n-1)x^{-1};$$

which, when $a' = b = c' = 0$, become

$$Q = \frac{a}{c}n^2.x^{2n-2}, \qquad P = \frac{b'n - c(n-1)}{cx};$$

which coincides with the first of the given equations by changing $2n - 2$ into n, and putting

$$A = \frac{b'(n+2) - cn}{2c}, \qquad B^2 = -\frac{a}{c}(\tfrac{1}{2}n+1)^2,$$

which requires that a and c be of opposite signs.

Case 2. Suppose $\quad z = e^{nx}$,

we find

$$Q = n^2 e^{2nx}.\frac{a' - ae^{nx}}{c' - ce^{nx}},$$

$$P = -ne^{nx}.\frac{b' - be^{nx}}{c' - ce^{nx}} - n;$$

which, when $a' = b = c' = 0$, become

$$Q = \frac{a}{c}n^2 e^{2nx}, \qquad P = \left(\frac{b'}{c} - 1\right)n;$$

and these coincide with the second given equation, by changing n into $\frac{1}{2}n$, and making

$$A = \tfrac{1}{2}\left(\frac{b'}{c} - 1\right)n, \qquad B^2 = -\frac{a}{4c}.\,n^2,$$

so that a and c must have opposite signs.

II. We will now consider the values of φ determined by equation (I), and the limits of the integration determined by the condition that

$$\Phi e^{-\theta z} = 0 \quad . \quad . \quad . \quad . \quad . \quad \text{(IV)}.$$

We will limit ourselves to the case involved in each of the preceding cases, that $a'=b=c'=0$, and $ac < 0$. Then (I) becomes

$$\frac{d\varphi}{\varphi}=\frac{(b' \quad 2c)\,\theta d\theta}{a+c\theta^2}, \text{ or, if } a=-cm^2,\ b'=2ch;\ \frac{d\varphi}{\varphi}=\frac{2\,(h-1)\,\theta d\theta}{\theta^2-m^2};$$

the integral of which gives

$$\varphi=(\theta^2-m^2)^{h-1},\ \text{or}=(m^2-\theta^2)^{h-1};$$

$$\Phi=c\varphi\,(\theta^2-m^2)=c\,(\theta^2-m^2)^h,\ \text{or}=-c\,(m^2-\theta^2)^h;$$

so that, when h is positive, (IV) is satisfied by $\theta=\pm m$; and, when z is positive, by $\theta=\infty$, when z is negative by $\theta=-\infty$.

We may then, for y', use the first value of φ, and the limits

$$\theta=\pm m \text{ to } \theta=\pm\infty,$$

the upper sign being used when z is positive, the lower when z is negative; and, for y'', we may use the second value of φ and the limits

$$\theta=-m \text{ to } \theta=+m.$$

SECOND SOLUTION. *By Dr. T. Strong, New-Brunswick.*

If in the first given equation we put $u=x^{n+2}=x^m$, and in the second $u=e^{nx}$, and suppose $du=$ const., they will become

$$\frac{d^2y}{du^2}+\left(1+\frac{A-1}{m}\right)\frac{dy}{udu}-\frac{B^2}{m^2}\cdot\frac{y}{u}=0 \quad . \quad . \quad . \quad (1),$$

$$\frac{d^2y}{du^2}+\left(1+\frac{A}{n}\right)\frac{dy}{udu}-\frac{B^2}{n^2}\cdot\frac{y}{u}=0 \quad . \quad . \quad . \quad (2);$$

which have the same form. The differential equation

$$\frac{d^2y}{du^2}+(2pq-q+1)\,\frac{dy}{udu}+q^2a^2b^2u^{2q-2}\,y=0 \quad . \quad (3),$$

is satisfied by $\quad y=c\int dv\,(a^2-v^2)^{p-1}\cos bu^q v \quad . \quad . \quad . \quad (4),$
the integral being taken between the limits $v=0$, $v=\pm a$, c being an arbitrary constant, and $p>0$; see Lacroix Calcul. Diff. et Int., Vol. III, p. 537. By comparing (1) and (3) we have $q=\frac{1}{2}$, $\frac{1}{4}a^2b^2=-B^2m^{-2}$;

$\therefore$ by assuming $a^2=1$, we get $b=\frac{2B}{m}\sqrt{-1}=b'\sqrt{-1}$; suppose also

$p+\frac{1}{2}=1+\frac{A-1}{m}$, or $p=\frac{2A+m-2}{2m}$, $p-1=\frac{2A-m-2}{2m}$; then we get

$$y=c\int dv(1-v^2)^{p-1}\cos b'u^{\frac{1}{2}}v\sqrt{-1}=c'\int dv(1-v^2)^{p-1}\left(e^{b'u^{\frac{1}{2}}v}+e^{-b'u^{\frac{1}{2}}v}\right).(5),$$

where $c=2c'$, and the integral is taken from $v=0$ to $v=1$. Again, if we put $v=v'\sqrt{-1}$, and $c=-c''\sqrt{-1}$, we have

$$y=c''\int dv'\,(1+v'^2)^{p-1}.\cos b'\,u^{\frac{1}{2}}v' \quad . \quad . \quad (6),$$

the limits of the integral being $v=0$, $v=\frac{\pi}{b'u^{\frac{1}{2}}}$; hence the complete value of y is the sum of the two values given by (5) and (6), c',c'' being the two arbitrary constants which the integral requires.

When $p<0$, the limits of the integral in (6) may also be $v'=0$ and $v'=\infty$, hence in this case the complete integral of (1) is

$$y=c'\int dv'\,(1+v'^2)^{p-1}\cos b'u^{\frac{1}{2}}v'+c''\int dv'\,(1+v'^2)^{p-1}\cos b'u^{\frac{1}{2}}v' \quad . (7),$$

the first integral being taken between the limits $v' = 0$, $v' = \frac{\pi}{b'u t}$ the second between the limits $v' = 0, v' = \infty$.

Again if we use the same values of q and a as before, and put $p = \frac{2A + n}{2n}$, $b' = \frac{2B}{n}$, then the complete value of y which satisfies (2) is the sum of the values in (5) and (6) when $p > 0$, or the value in (7) when $p < 0$.

Remark. See Mec. Cel., Vol. IV, supp. on capillary attraction, page 60, eq. (5) which is of the form

$$\frac{d^2 z}{du^2} + \frac{dz}{udu} - 2\left(a'z + \frac{1}{b''}\right) = 0,$$

which, by putting $z + \frac{1}{a'b''} = y$ becomes

$$\frac{d^2 y}{du^2} + \frac{dy}{udu} - 2a'y = 0,$$

and if this be compared with (4) it gives $q=1, p = \frac{1}{2}, a^2=1, b = \sqrt{-2a'}$, or $b' = \sqrt{2a'}$; and if these values be written in (5) and (6), the sum of these values will give the complete integral of La Place's equation which he failed to find; nor do I perceive that the deficiency has been supplied in Dr. Bowditch's Commentary. The partial integral which he finds, and which is equivalent to (5), is however sufficient for the particular purpose he had in view.

——— The following equations, which are frequently met with in Physico-Mathematical researches, will be found to be immediately dependant on each other, and their solutions, as well as those of many others, are comprehended in Prof. Peirce's Analysis :—

$$\frac{d^2 y}{dx^2} + \frac{A}{x} \cdot \frac{dy}{dx} - B^2 x^n y = 0,$$

$$\frac{d^2 y}{dx^2} + A \cdot \frac{dy}{dx} - B^2 e^{nx} y = 0,$$

$$\frac{d^2 y}{dx^2} - (A^2 + B^2 e^{nx})\, y = 0,$$

$$\frac{d^2 y}{dx^2} + \frac{A}{x} \cdot \frac{dy}{dx} - B^2 y = 0.$$

$$\frac{d^2 y}{dx^2} + \frac{A}{x}\, \frac{dy}{dx} - \frac{B^2}{x} \cdot y = 0,$$

$$\frac{d^2 y}{dx^2} + \left(\frac{A}{x^2} - B^2\right) y = 0,$$

$$\frac{d^2 y}{dx^2} - B^2 x^n y = 0;$$

and on the last depends the equation of Ricatti.

List of Contributors, and of Questions answered by each. The figures refer to the number of the questions, as marked in Number VI., Article XXV.

Prof. C. Avery, Hamilton College, N. Y., ans. all the questions.
P. Barton, Jun., Esperance, N. Y., ans. 1, 2, 3, 4.
B. Birdsall, Clinton Liberal Institute, N. Y., ans. 1 to 12, 15.
Prof. M. Catlin, Hamilton College, Clinton, N. Y., ans. all the questions.
E. H. Delafield, St. Paul's College, N. Y., ans. 3.
Engineer, ans. 1.
D. Kirkwood, York, Pa., ans. 3.
Wm. Lenhart, York, Pa., ans. 2, 3, 8.
L., Murray Co., Geo., ans. 1, 2, 3, 4, 9.
J. F. Macully, Teacher of Mathematics, New-York, ans. 1 to 10.
Omicron, Jun., Chapel Hill, N. C., ans. 3.
Prof. B. Peirce, Harvard University, Cambridge, ans. all the questions.
Prof. G. R. Perkins, Utica Academy, N. Y., ans. all the questions.
P., ans. 3, 10.
ψ, ans. 5, 13.
Prof. T. Strong, L. L. D., New Brunswick, N. J., ans. all the questions.

*** All communications for Number IX, which will be published on the first day of May, 1840, must be post paid, addressed to the Editor, *College Point*, N. Y., and must arrive before the first of February, 1840. New Questions must be accompanied with their solutions.

The Editor begs to thank Mr. Spiller for a copy of his beautiful translation of Sturm's Theorem, and its demonstration.

A correspondent is right in supposing that the results of Mr. Young are erroneous in the problem referred to.

ARTICLE V.

NEW QUESTIONS TO BE ANSWERED IN NUMBER X.

Their solutions must arrive before August 1st, 1840,

(128). QUESTION I. *By P.*

Transform the value of x

$$x = a \sin \varphi + b \cos \varphi.$$

into a real form, when a and φ become imaginary.

(129). QUESTION II. *By P.*

Find all the values of

$$(1 + \sqrt{-1})^{\frac{1}{7}} + (1 - \sqrt{-1})^{\frac{1}{7}}.$$

(130). QUESTION III. *By Wm. Lenhart, Esq., York, Pa.*

Find x, y, x', y', so that

$$x^3 + y^3 + xy \text{ may be a square,}$$
$$x'^3 + y'^3 + x'y' \text{ may be a cube.}$$

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(131). QUESTION IV. *By Mr. P. Barton, Jun., Esperance, N. Y.*

Let $x + y + z - 10 = 0$, be the equation of a plane, (2, 4, 4) the centre of a circle in that plane, whose radius is 6; to find the least distance from the point (1, 2, 3) to the periphery of that circle.

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(132). QUESTION V. *By J. F. Macully, Esq. New-York.*

On a given regular polygon, as base, to construct a right pyramid, which shall have each of the solid angles at the base equal to m times the solid angle at the vertex.

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(133). QUESTION VI. *By Prof. W. McCartney, Easton, Pa.*

The centre of a circle moves along a given straight line, while its radius varies as the square of the distance of the centre from a given point in that line. It is required to find the line to which the circle is always a tangent.

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(134). QUESTION VII. *By J. F. Macully, Esq.*

Find the sum of n terms of the series

$$\frac{\sin^7\theta}{\cos\theta\cos2\theta} + \frac{8\sin^7\frac{1}{2}\theta}{\cos\frac{1}{2}\theta\cos\theta} + \frac{8^2\sin^7\frac{1}{4}\theta}{\cos\frac{1}{4}\theta\cos\frac{1}{2}\theta} + \&c.$$

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(135). QUESTION VIII. *By M. Terquem.*

To find the number of normals to a given algebraic surface of the degree m, that can be drawn through a given point.

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(136). QUESTION IX. *By M. Terquem.*

To find the locus of the points from which the sum of the squares of all the normals that can be drawn to a given surface of the second degree is a given constant quantity.

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(137). QUESTION X. *By ——.*

If P_x, P_{x+1}, P_{x+2} represent the consecutive terms of a series of integers which are at once polygonal numbers of both the m^{th} and n^{th} orders, they will be connected by the equation

$$P_{x+2} - 2aP_{x+1} + P_x = b;$$

where a and b are constants depending on the numbers m and n (see question (44), Math. Misc., Vol. I). It is required to express P_x in terms of x.

(138). QUESTION XI. *From the Dublin Problems.*

If any two polygons, having the same number of sides, be circumscribed about a given ellipse, so that the points of contact may be the middle points of each side; then, of these two polygons,

1°. Their areas are equal.

2°. The sum of the squares of their sides are equal.

3°. The sum of the squares of the semidiameters drawn to the points of contact of the sides, are equal.

4°. The sum of the squares of the lines drawn from the centre to the angular points of the polygons are equal.

(139). QUESTION XII. *By* Δ.

Find the quickest path between two given points on the surface of a sphere; the force residing in another given point of the surface, and varying as some function of the distance.

(140). QUESTION XIII. *By M. Poisson.*

Ans. in Cambr. p. 6[illegible]

Three players A, B, C, play, two by two, a series of games; each new game is played by the one who has won the preceding game, with the one who did not play, and the two who play the first game is determined by lot. The party is terminated when one of the three players has won two successive games, and this player gains the party. It is required to determine, for each player, the probability he has of winning the party, having given his chance of winning a game from either of his opponents.

(141). QUESTION XIV. *By Investigator.*

M. Jacobi has shown that there exists a class of homogeneous liquid ellipsoids with three unequal axes, susceptible of equilibrium when revolving round one of these axes, with a centrifugal force. It is required to determine "what are the precise limits within which this extension of the problem is possible."

Mr. Ivory and M. Liouville arrive at contradictory results on this subject.

(142). QUESTION XV. *By Prof. B. Peirce, Harvard University.*

C. M p. [illegible]

Professor ——— found it impossible to spin a top upon a hard steel floor, the point being perfectly sharp like that of a needle; while he found no difficulty in spinning one whose point was blunted and nearly hemispherical. Can this difference be explained by analysis?

Prof. P. desires it to be stated, that he has not attempted the solution of this question.

ARTICLE VI.

SUMMATION OF TRIGONOMETRICAL SERIES.

By J. F. Macully, Esq., New-York.

LITTLE has yet been done in the Inverse method of Finite Differences. Herschel's examples contain the most complete collection I have seen; and in these there is no approach to any classification of forms, beyond those of the most simple Algebraical and Circular Functions. It would seem that, like all Converse methods of operation, examples of the direct process must be multiplied to a great extent, before the results can be grouped in such a way as to lead to a direct knowledge of the primitive function from which a given Difference is derived. My aim in drawing up the following article, has been to contribute my quota to such a collection of examples. I have only inserted a few of the more remarkable results under the different classes, finding that to insert a greater number, as I originally designed, would have encroached too much on the limits of the Miscellany. The Classes marked (IV.), (V.), (VI.), (VII.), (VIII.) are particularly deficient, while they comprehend many results worth noting. If what I have done meets with the approbation of the readers of the Miscellany, I may at a future time add other examples together with forms derived from the combination of two or more of these Classes. I have added a few examples of series and products, the sums of which are derived from these examples.

Examples of the finite Differences of Trigonometrical Functions.

(I). $$\Delta .\ h^x \sin k^x\theta = h^{x+1} \sin k^{x+1}\theta - h^x \sin k^x\theta$$
$$= h^x(h \sin k^{x+1}\theta - \sin k^x\theta),$$

where h, k, θ are constant, and $\Delta x = 1$.

Example 1. Let $k = 2$, then since

$$\sin 2^{x+1}\theta = \sin 2 \,.\, 2^x\theta = 2 \sin 2^x\theta \cos 2^x\theta,$$

(*a*) $$\Delta .\ h^x \sin 2^x\theta = h^x(h \sin 2^{x+1}\theta - \sin 2^x\theta)$$
$$= h^x \sin 2^x\theta(2h \cos 2^x\theta - 1).$$

Thus, 1. $$\Delta \sin 2^x\theta = \sin 2^{x+1}\theta - \sin 2^x\theta$$
$$= 2 \sin 2^{x-1}\theta \cos 2^{x-1}.3\theta,$$

2. $$\Delta 2^{-x} \sin 2^x\theta = 2^{-x} \sin 2^x\theta(\cos 2^x\theta - 1)$$
$$= -2^{-x+1} \sin 2^x\theta \sin^2 2^{x-1}\theta.$$

Example 2. Let $k = \frac{1}{2}$, then

(*b*) $$\Delta . h^x \sin 2^{-x}\theta = h^x(h \sin 2^{-x-1}\theta - \sin 2^{-x}\theta)$$
$$= h^x \sin 2^{-x-1}\theta(h - 2 \cos 2^{-x-1}\theta).$$

Thus, 3. $$\Delta \sin 2^{-x}\theta = \sin 2^{-x-1}\theta - \sin 2^{-x}\theta$$
$$= -2 \sin 2^{-x-2}\theta \cos 2^{-x-2}.3\theta,$$

4. $$\Delta . 2^x \sin 2^{-x}\theta = 2^{x+1} \sin 2^{-x-1}\theta(1 - \cos 2^{-x-1}\theta)$$
$$= 2^{x+2} \sin 2^{-x-1}\theta \sin^2 2^{-x-2}\theta.$$

Example 3. Let $k = 3$, then, since

$$\sin 3^{x+1}\theta = \sin 3 \,.\, 3^x\theta$$
$$= \sin 3^x\theta(2 \cos 3^x.2\theta + 1),$$

(*c*) $$\Delta .\ h^x \sin 3^x\theta = h^x(h \sin 3^{x+1}\theta - \sin 3^x\theta)$$
$$= h^x \sin 3^x\theta(2h \cos 3^x \,.\, 2\theta + h - 1).$$

Thus, 5. $\Delta \sin 3^x\theta = 2\sin 3^x\theta \cos 3^x . 2\theta,$

6. $\Delta\, 2^{-x} \sin 3^x\theta = 2^{-x-1} \sin 3^x\theta(2\cos 3^x . 2\theta - 1)$

$$= 2^{-x-1} \sin 3^x\theta . \frac{\cos 3^{x+1}\theta}{\cos 3^x\theta}$$

$$= 2^{-x-1} \tan 3^x\theta \cos 3^{x+1}\theta,$$

7. $\Delta.(-1)^x \sin 3^x\theta = (-1)^{x+1}. 2\sin 3^x\theta(\cos 3^x . 2\theta + 1)$

$$= (-1)^{x+1}. 4\sin 3^x\theta \cos^2 3^x\theta$$

$$= (-1)^{x+1}. 2\sin 3^x. 2\theta \cos 3^x\theta$$

Example 4. Let $k = \frac{1}{3}$, then

(*d*) $\Delta. h^x \sin 3^{-x}\theta = h^x (h \sin 3^{-x-1}\theta - \sin 3^{-x}\theta)$

$$= h^x \sin 3^{-x-1}\theta(h - 1 - 2\cos 3^{-x-1}. 2\theta).$$

Thus 8. $\Delta \sin 3^{-x}\theta = -2\sin 3^{-x-1}. \theta \cos 3^{-x-1}. 2\theta,$

9. $\Delta (-1)^x \sin 3^{-x}\theta = -4 (-1)^x \sin 3^{-x-1}\theta \cos^2 3^{-x-1}\theta,$

10. $\Delta\, 2^x \sin 3^{-x}\theta = 2^x \sin 3^{-x-1}\theta\, (1 - 2\cos 3^{-x-1}. 2\theta)$

$$= -2^x \cos 3^{-x}\theta \tan 3^{-x-1}\theta,$$

11. $\Delta\, 3^x \sin 3^{-x}\theta = 4. 3^x \sin^3 3^{-x-1}\theta.$

Example 5. Let $h = 1$, $k = \frac{1}{5}$,

12. $\Delta \sin 5^{-x}\theta = \sin 5^{-x-1}\theta - \sin 5^{-x}\theta$

$$= -2\sin\tfrac{1}{2}\,(5^{-x} - 5^{-x-1})\theta \cos\tfrac{1}{2}\,(5^{-x} + 5^{-x-1})\theta$$

$$= -2\sin 2. 5^{-x-1}\theta \cos 3. 5^{-x-1}\theta.$$

(II.) $\Delta\, h^x \cos k^x\theta = h^x (h \cos k^{x+1}\theta - \cos k^x\theta).$

Example 1. Let $k = 2$, then, since

$$\cos 2^{x+1}\theta = \cos 2. 2^x\theta = 2\cos^2 2^x\theta - 1,$$

(*e*) $\Delta. h^x \cos 2^x\theta = h^x(2h \cos^2 2^x\theta - \cos 2^x\theta - h).$

Thus; 13. $\Delta \cos 2^x\theta = 2\sin 3. 2^{x-1}\theta \sin 2^{x-1}\theta,$

14. $\Delta (-1)^x \cos 2^x\theta = (-1)^{x+1}. 2\cos 3. 2^{x-1}\theta \cos 2^{x-1}\theta.$

Example 2. Let $k = \frac{1}{2}$, then

(*f*) $\Delta. h^x \cos 2^{-x}\theta = h^x (h \cos 2^{-x-1}\theta - \cos 2^{-x}\theta)$

$$= h^x (h \cos 2^{-x-1}\theta - 2\cos^2 2^{-x-1}\theta + 1).$$

Thus, 15. $\Delta \cos 2^{-x}\theta = 2\sin 2^{-x-2}\theta \sin 2^{-x-2}. 3\theta,$

16. $\Delta.(-1)^x \cos 2^{-x}\theta = (-1)^{x+1}. 2\cos 2^{-x-2}\theta \cos 2^{-x-2}. 3\theta,$

17. $\Delta. 2^x \cos 2^{-x}\theta = 2^x(1 + 4\cos 2^{-x-1}\theta \sin^2 2^{-x-2}\theta).$

Example 3. Let $k = 3$, then

(*g*) $\Delta. h^x \cos 3^x\theta = h^x (h \cos 3^{x+1}\theta - \cos 3^x\theta)$

$$= h^x \cos 3^x\theta\, (2h \cos 2. 3^x\theta - h - 1).$$

Thus; 18. $\Delta. \cos 3^x\theta = -4\sin^2 3^x\theta \cos 3^x\theta,$

19. $\Delta. (-1)^x \cos 3^x\theta = 2 (-1)^{x+1} \cos 3^x\theta \cos 2. 3^x\theta,$

20. $\Delta\, 2^x \cos 3^x\theta = 2^x \cos 3^x\theta\, (1 - 8\sin^2 3^x\theta),$

21. $\Delta.(-2)^{-x} \cos 3^x\theta = (-2)^{-x-1} \cot 3^x\theta \sin 3^{x+1}\theta,$

22. $\Delta.(-3)^{-x} \cos 3^x\theta = (-3)^{-x-1}. 4\cos 3^x\theta \sin^2 3^x\theta.$

Example 4. Let $k = \frac{1}{3}$, then

(*h*) $\Delta. h^x \cos 3^{-x}\theta = h^x \cos 3^{-x-1}\theta\, (h + 1 - 2\cos 2. 3^{-x-1}\theta).$

Thus, 23. $\Delta \cos 3^{-x}\theta = 4\cos 3^{-x-1}\theta \sin^2 3^{-x-1}\theta$

$$= 2\sin 3^{-x-1}\theta \sin 3^{-x-1}. 2\theta,$$

24. $\Delta (-1)^x \cos 3^{-x}\theta = (-1)^{x+1}. 2\cos 3^{-x-1}\theta \cos 2. 3^{-x-1}\theta,$

25. $\Delta (-2)^x \cos 3^{-x}\theta = -(-2)^x \sin 3^{-x}\theta \cot 3^{-x-1}\theta,$

26. $\Delta (-3)^x \cos 3^{-x}\theta = -(-3)^x. 4\cos^3 3^{-x-1}\theta.$

(III). $$\Delta.\frac{1}{k^x \sin k^x\theta} = -\frac{\Delta.\, k^x \sin k^x\theta}{k^{2x+1}\sin k^x\theta \sin k^{x+1}\theta}.$$

Thus, 27. $$\Delta.\frac{1}{\sin 2^x\theta} = \frac{-2\sin 2^{x-1}\theta \cos 2^{x-1}.3\theta}{\sin 2^x\theta \sin 2^{x+1}\theta} = -\frac{\cos 2^{x-1}.\,3\theta}{\cos 2^{x-1}\theta \sin 2^{x+1}\theta},$$

28. $$\Delta.\frac{2^x}{\sin 2^x\theta} = -\frac{2^x \tan 2^{x-1}\theta}{\cos 2^x\theta} = \frac{2^{x+1}\sin^2 2^{x-1}\theta}{\sin 2^{x+1}\theta},$$

29. $$\Delta.\frac{1}{\sin 2^{-x}\theta} = \frac{2\sin 2^{-x-2}\theta \cos 2^{-x-2}.\,3\theta}{\sin 2^{-x}\theta \sin 2^{-x-1}\theta} = \frac{\cos 2^{-x-2}.\,3\theta}{\sin 2^{-x}\theta \cos 2^{-x-2}\theta}.$$

30. $$\Delta.\frac{1}{2^x \sin 2^{-x}\theta} = \frac{-\sin^2 2^{-x-2}\theta}{2^{x-1}\sin 2^{-x}\theta},$$

31. $$\Delta.\frac{1}{\sin 3^x\theta} = -\frac{2\cos 3^x.\,2\theta}{\sin 3^{x+1}\theta},$$

32. $$\Delta.\frac{(-1)^x}{\sin 3^x\theta} = \frac{4.\,(-1)^{x+1}\cos^2 3^x\theta}{\sin 3^{x+1}\theta},$$

33. $$\Delta.\frac{2^x}{\sin 3^x\theta} = -2^x \sec 3^x\theta \cot 3^{x+1}\theta,$$

34. $$\Delta.\frac{1}{2^x \sin 3^{-x}\theta} = \frac{1}{2^{x+1}}.\cot 3^{-x}\theta \sec 3^{-x-1}\theta,$$

35. $$\Delta.\frac{1}{3^x \sin 3^{-x}\theta} = -\frac{4}{3^{x+1}}.\frac{\sin^2 3^{-x-1}\theta}{\sin 3^{-x}\theta},$$

36. $$\Delta.\frac{1}{\sin 5^{-x}\theta} = \frac{4\cos 5^{-x-1}\theta \cos 3.\,5^{-x-1}\theta}{\sin 5^{-x}\theta}.$$

(IV.) $$\Delta.\frac{1}{k^x \cos k^x\theta} = \frac{-\Delta.\, k^x \cos k^{x+1}\theta}{k^{2x+1}\cos k^x\theta \cos k^{x+1}\theta}.$$

Thus, 37. $$\Delta.\frac{1}{\cos 3^{-x}\theta} = -\frac{4\sin^2 3^{-x-1}\theta}{\cos 3^{-x}\theta},$$

38. $$\Delta.\frac{1}{(-3)^x \cos 3^{-x}\theta} = \frac{4\cos^2 3^{-x-1}\theta}{(-3)^{x+1}\cos 3^{-x}\theta}.$$

(V.) $$\Delta.\frac{1}{k^x \sin^2 k^x\theta} = \frac{\Delta.\, k^x \cos k^x.\,2\theta - \Delta k^x}{2k^{2x+1}\sin^2 k^x\theta \sin^2 k^{x+1}\theta}.$$

Thus, 39. $$\Delta.\frac{1}{\sin^2 3^{-x}\theta} = \frac{4\cos 3^{-x-1}.\,2\theta \sin 3^{-x-1}.\,2\theta}{2\sin^2 3^{-x}\theta \sin^2 3^{-x-1}\theta}$$
$$= \frac{8\cos 3^{-x-1}.\,2\theta \cos^2 3^{-x-1}\theta}{\sin^2 3^{-x}\theta},$$

40. $$\Delta.\frac{1}{2^x \sin^2 2^{-x}\theta} = \frac{\cos 2^{-x}\theta}{2^x \sin^2 2^{-x}\theta}.$$

(VI.) $$\Delta.\frac{1}{k^x \cos^2 k^x\theta} = \frac{-\Delta.\, k^x \cos k^x.\,2\theta - \Delta k^x}{2k^{2x+1}\cos^2 k^x\theta \cos^2 k^{x+1}\theta}.$$

Thus, 41. $$\Delta.\frac{1}{\cos^2 2^{-x}\theta} = \frac{-\sin 2^{-x-1}\theta \sin 2^{-x-1}.\,3\theta}{\cos^2 2^{-x}\theta \cos^2 2^{-x-1}.\,\theta}$$
$$= \frac{-16\sin^3 2^{-x-1}\theta \sin 2^{-x-1}.\,3\theta}{\sin^2 2^{-x+1}\theta}$$

42. $\Delta . \dfrac{1}{\cos^2 4^{-x}\theta} = \dfrac{-\sin 4^{-x-1} . 3\theta \sin 4^{-x-1} . 5\theta}{\cos 4^{-x}\theta \cos 4^{-x-1}\theta}$

(VII.) $\Delta . \; h^x \cot k^x\theta = h^x . \dfrac{(h-1)\sin k^x (k+1)\,\theta - (h+1)\sin k^x (k-1)\,\theta}{2 \sin k^x\theta \sin k^{x+1}\theta}.$

Thus, 43. $\Delta \cot k^x\theta = -\dfrac{\sin k^x (k-1)\,\theta}{\sin k^x\theta \sin k^{x+1}\theta},$

44. $\Delta . 3^x \cot 2^x\theta = 3^x \dfrac{\cos 2^x . 3\theta}{\cos 2^x\theta \sin 2^{x+1}\theta},$

45. $\Delta \cot 3^{-x}\theta = \dfrac{\sin 3^{-x-1} . 2\theta}{\sin 3^{-x}\theta \sin 3^{-x-1}\theta},$

$$= \frac{2 \cos 3^{-x-1}\theta}{\sin 3^{-x}\theta},$$

46. $\Delta . \; 3^x \cot 3^{-x}\theta = 3^x \dfrac{2 \sin 3^{-x-1} . 4\theta + 4 \sin 3^{-x-1} . 2\theta}{2 \sin 3^{-x}\theta \sin 3^{-x-1}\theta}$

$$= 3^x . \frac{2 \sin 3^{-x-1} . 2\theta \, (1 + \cos 3^{-x-1} . 2\theta)}{\sin 3^{-x}\theta \sin 3^{-x-1}\theta}$$

$$= 3^x . \frac{8 \cos^3 3^{-x-1}\theta}{\sin 3^{-x}\theta},$$

47. $\Delta . \; 3^{-x} \cot 3^{-x}\theta = \dfrac{4}{3^{x+1}} . \dfrac{\sin 3^{-x-1} . 2\theta \sin 3^{-x-1}\theta}{\sin 3^{-x}\theta}.$

(VIII.) $\Delta . \, h^x \tan k^x\theta = h^x . \dfrac{(h-1)\sin k^x (k+1)\theta + (h+1) \sin k^x (k-1)\,\theta}{2 \cos k^x\theta \cos k^{x+1}\theta}$

$$= - h^{2x+1} \tan k^x\theta \tan k^{x+1}\theta \;\; \Delta . \; h^{-x} \cot k^x\theta.$$

Examples of Trigonometrical Series.

If u_x represent the x^{th} term of a series, s_x the sum of x terms, s the sum of an infinite number of terms, Σ the symbol of finite integration, and c a constant quantity,

$$s_x = u_1 + u_2 + u_3 + \; \ldots \; + u_x,$$
$$s_{x+1} = u_1 + u_2 + u_3 + \; \ldots \; + u_x + u_{x+1} \; ;$$
$$s_{x+1} - s_x = \Delta \, s_x = u_{x+1},$$
and $s_x = c + \Sigma \, u_{x+1}$.

Example 1. Let $s_x = \sin \theta \sin^2 \frac{1}{2}\theta + 2 \sin \frac{1}{2}\theta \sin^2 \frac{1}{4}\theta + \ldots$

$$+ 2^{x-1} \sin \frac{\theta}{2^{x-1}} \sin^2 \frac{\theta}{2^x}.$$

$$u_{x+1} = 2^x \sin 2^{-x}\theta \sin^2 2^{-x-1}\theta,$$

and by equation (4)

$$s_x = c + \Sigma . 2^x \sin 2^{-x} \sin^2 2^{-x-1}\theta$$
$$= c + 2^{x-3} \sin 2^{-x+1}\theta,$$
$$s_0 = 0 = c + \tfrac{1}{4} \sin 2\theta,$$
$$s_x = 2^{x-2} \sin 2^{-x+1}\theta - \tfrac{1}{4} \sin 2\theta.$$

When $x = \infty$, the angle $2^{-x+1}\theta$ becomes indefinitely small, and its sine becomes equal to the arc, so that, then,

$$2^{x-2} \sin 2^{-x+1}\theta = 2^{x-2} \times 2^{-x+1}\theta = \tfrac{1}{2}\theta,$$
$$s = \tfrac{1}{2}\theta - \tfrac{1}{4} \sin 2\theta.$$

Example 2. Let $s_x = \tan\theta\cos 3\theta + \frac{1}{2}\tan 3\theta\cos 9\theta + \ldots . + 2^{-x+1}\tan 3^{x-1}\theta\cos 3^x\theta$,

$$u_{x+1} = 2^{-x}\tan 3^x\theta\cos 3^{x+1}\theta$$

and (eq. 6), $s_x = c + \Sigma\, 2^{-x}\tan 3^x\theta\cos 3^{x+1}\theta$

$$= c + 2^{-x+1}\sin 3^x\theta$$

$$s_0 = 0 = c + 2\sin\theta$$

$$s_x = 2(2^{-x}\sin 3^x\theta - \sin\theta).$$

Example 3. Let $s_x = \sin\theta\cos^2\theta - \sin 3\theta\cos^2 3\theta + \ldots . . (-1)^{x-1}\sin 3^{x-1}\theta\cos^2 3^{x-1}\theta$.

$$u_{x+1} = (-1)^x\sin 3^x\theta\cos^2 3^x\theta,$$

and (eq. 7), $s_x = c - \frac{1}{4}(-1)^x\sin 3^x\theta$;

$$s_0 = 0 = c - \tfrac{1}{4}\sin\theta,$$

$$s_x = \tfrac{1}{4}\sin\theta - \tfrac{1}{4}(-1)^x\sin 3^x\theta.$$

Example 4. Let $s_x = \sin\frac{1}{3}\theta\cos\frac{1}{2}\theta + \sin\frac{1}{15}\theta\cos\frac{1}{10}\theta + \ldots . + \sin 5^{-x+1}.\frac{1}{3}\theta\cos 5^{-x+1}.\frac{1}{2}\theta$

$$u_{x+1} = \sin 5^{-x}.\tfrac{1}{3}\theta\cos 5^{-x}.\tfrac{1}{2}\theta;$$

or, if $\theta = \frac{6}{5}\theta'$, $u_{x+1} = \sin 5^{-x-1}.2\theta'\cos 5^{-x-1}.2\theta'$;

and (eq. 12) $s_x = c - \frac{1}{2}\sin 5^{-x}.\theta' = c - \frac{1}{2}\sin 5^{-x+1}.\frac{1}{6}\theta$,

$$s_0 = 0 = c - \tfrac{1}{2}\sin\tfrac{5}{6}\theta,$$

$$s_x = \tfrac{1}{2}\sin\tfrac{5}{6}\theta - \tfrac{1}{2}\sin 5^{-x+1}.\tfrac{1}{6}\theta;$$

and $s = \frac{1}{2}\sin\frac{5}{6}\theta$.

Example 5. Let $s_x = \cos\theta\sin^2\frac{1}{2}\theta + 2\cos\frac{1}{2}\theta\sin^2\frac{1}{4}\theta + \ldots . , + 2^{x-1}\cos\dfrac{\theta}{2^{x-1}}\sin^2\dfrac{\theta}{2^x}$

$$u_{x+1} = 2^x\cos 2^{-x}\theta\sin^2 2^{-x-1}\theta$$

$$= \tfrac{1}{2}\Delta.\, 2^{x-1}\cos 2^{-x+1}\theta - \Delta 2^{x-2}, \qquad \text{(eq. 17)}$$

$$s_x = c + 2^{x-2}\cos 2^{-x+1}\theta - 2^{x-2}$$

$$= c - 2^{x-1}\sin^2 2^{-x}\theta;$$

$$s_0 = c - \tfrac{1}{2}\sin^2\theta = 0,$$

$$s_x = \tfrac{1}{2}(\sin^2\theta - 2^x\sin^2 2^{-x}\theta).$$

$$s = \tfrac{1}{2}\sin^2\theta.$$

Example 6. Let $s_x = \cot\theta\sec\frac{1}{3}\theta + \frac{1}{2}\cot\frac{1}{3}\theta\sec\frac{1}{9}\theta + \ldots . . + \dfrac{1}{2^{x-1}}\cot\dfrac{\theta}{3^{x-1}}\sec\dfrac{\theta}{3^x}$.

$$u_{x+1} = 2^{-x}\cot 3^{-x}\theta\sec 3^{-x-1}\theta,$$

and (eq. 34) $s_x = c + 2^{-x+1}\operatorname{cosec} 3^{-x}\theta$;

$$s_0 = c + 2\operatorname{cosec}\theta = 0,$$

$$s_x = 2(2^{-x}\operatorname{cosec} 3^{-x}\theta - \operatorname{cosec}\theta).$$

Example 7. Let $s_x = \dfrac{\sin^2\frac{1}{3}\theta}{\sin\theta} + \dfrac{1}{3}.\dfrac{\sin^2\frac{1}{9}\theta}{\sin\frac{1}{3}\theta} + \ldots . + \dfrac{1}{3^{x-1}}.\dfrac{\sin^2 3^{-x}\theta}{\sin 3^{-x+1}\theta}$.

$$u_{x+1} = \frac{1}{3^x}.\frac{\sin^2 3^{-x-1}\theta}{\sin 3^{-x}\theta},$$

and (eq. 35). $s_x = c - \dfrac{1}{4.3^{x-1}\sin 3^{-x}\theta}$;

$$s_0 = c - \frac{3}{4\sin\theta} = 0,$$

$$s_x = \tfrac{3}{4}\left(\frac{1}{\sin\theta} - \frac{1}{3^x\sin 3^{-x}\theta}\right).$$

$$s = \tfrac{3}{4}\left(\frac{1}{\sin\theta} - \frac{1}{\theta}\right).$$

Ex. 8. Let $$s_x = \frac{\cos\frac{1}{5}\theta\cos\frac{3}{5}\theta}{\sin\theta} + \frac{\cos\frac{1}{25}\theta\cos\frac{3}{25}\theta}{\sin\frac{1}{5}\theta} + \ldots + \frac{\cos 5^{-x}\theta\cos 3.5^{-x}\theta}{\sin 5^{-x+1}\theta}.$$

$$u_{x+1} = \frac{\cos 5^{-x-1}\theta\cos 3.5^{-x-1}\theta}{\sin 5^{-x}\theta};$$

and (eq. 36), $$s_x = c + \frac{1}{4\sin 5^{-x}\theta}$$
$$= \tfrac{1}{4}\left(\frac{1}{\sin 5^{-x}\theta} - \frac{1}{\sin\theta}\right).$$

Ex. 9. Let $$s_x = \frac{\cos^2\frac{1}{3}\theta}{\cos\theta} - \tfrac{1}{3}.\frac{\cos^2\frac{1}{9}\theta}{\cos\frac{1}{3}\theta} + \ldots\ldots + (-\tfrac{1}{3})^{x-1}.\frac{\cos^2 3^{-x}\theta}{\cos 3^{-x+1}\theta}.$$

$$u_{x+1} = \frac{1}{(-3)^x}.\frac{\cos^2 3^{-x-1}\theta}{\cos 3^{-x}\theta},$$

and (eq. 38), $$s_x = c + \frac{1}{4.(-3)^{x-1}\cos 3^{-x}\theta}$$
$$= \tfrac{3}{4}\left(\frac{1}{\cos\theta} - \frac{1}{(-3)^x\cos 3^{-x}\theta}\right). \qquad s = \tfrac{3}{4}\sec\theta$$

Ex. 10. Let $$s_x = \frac{\sin^2\frac{1}{3}\theta}{\cos\theta} + \frac{\sin^2\frac{1}{9}\theta}{\cos\frac{1}{3}\theta} + \ldots + \frac{\sin^2 3^{-x}\theta}{\cos 3^{-x+1}\theta}.$$

$$u_{x+1} = \frac{\sin^2 3^{-x-1}\theta}{\cos 3^{-x}\theta},$$

and (eq. 37.) $$s_x = c - \frac{1}{4\cos 3^{-x}\theta}$$
$$= \tfrac{1}{4}\left(\frac{1}{\cos\theta} - \frac{1}{\cos 3^{-x}\theta}\right). \qquad s = \tfrac{1}{4}(\sec\theta - 1).$$

Ex. 11. Let $$s_x = \frac{\cos^2\frac{1}{3}\theta\cos\frac{2}{3}\theta}{\sin^2\theta} + \frac{\cos^2\frac{1}{9}\theta\cos\frac{2}{9}\theta}{\sin^2\frac{1}{3}\theta} + \ldots\ldots + \frac{\cos^2 3^{-x}\theta\cos 3^{-x}.2\theta}{\sin^2 3^{-x+1}\theta}.$$

$$u_{x+1} = \frac{\cos^2 3^{-x-1}\theta\cos 3^{-x-1}.2\theta}{\sin^2 3^{-x}\theta},$$

and (eq. 39) $$s_x = c + \tfrac{1}{8}\operatorname{cosec}^2 3^{-x}\theta$$
$$= \tfrac{1}{8}(\operatorname{cosec}^2 3^{-x}\theta - \operatorname{cosec}^2\theta).$$

Ex. 12. Let $$s_x = \frac{\cos\theta}{\sin^2\theta} + \tfrac{1}{2}.\frac{\cos\frac{1}{2}\theta}{\sin^2\frac{1}{2}\theta} + \ldots\ldots + \frac{1}{2^{x-1}}.\frac{\cos 2^{-x+1}\theta}{\sin^2 2^{-x-1}\theta}.$$

$$u_{x+1} = \frac{\cos 2^{-x}\theta}{2^x\sin^2 2^{-x}\theta},$$

and (eq. 40) $$s_x = c + \frac{1}{2^x\sin^2 2^{-x}\theta}$$
$$= \frac{1}{2^x\sin^2 2^{-x}\theta} - \frac{1}{\sin^2\theta}.$$

Ex. 13. Let $s_x = \frac{\sin^3 \frac{1}{2}\theta \sin \frac{3}{2}\theta}{\sin^2 \frac{1}{2}\theta} + \frac{\sin^3 \frac{1}{4}\theta \sin \frac{3}{4}\theta}{\sin^2 2\theta} + \ldots\ldots + \frac{\sin^3 2^{-x}\theta \sin 2^{-x}.3\theta}{\sin^2 2^{-x+2}\theta}$

$$u_{x+1} = \frac{\sin^3 2^{-x-1}\theta \sin 2^{-x-1}.3\theta}{\sin^2 2^{-x+1}\theta},$$

and (eq. 41) $s_x = c - \frac{1}{16}\sec^2 2^{-x}\theta$

$= \frac{1}{16}(\sec^2\theta - \sec^2 2^{-x}\theta)$ $\qquad s = \frac{1}{16}\tan^2\theta$

Ex. 14. Let $s_x = \frac{\sin\frac{3}{4}\theta \sin\frac{5}{4}\theta}{\cos^2\theta\cos^2\frac{1}{4}\theta} + \frac{\sin\frac{3}{16}\theta\sin\frac{5}{16}\theta}{\cos^2\frac{1}{4}\theta\cos^2\frac{1}{16}\theta} + \ldots + \frac{\sin 4^{-x}.3\theta \sin 4^{-x}.5\theta}{\cos^2 4^{-x+1}\theta\cos^2 4^{-x}\theta}$.

$$u_{x+1} = \frac{\sin 4^{-x-1}.3\theta \sin 4^{-x-1}.5\theta}{\cos^2 4^{-x}\theta\cos^2 4^{-x-1}\theta},$$

and (eq. 42) $s_x = c - \sec^2 4^{-x}\theta$

$= \sec^2\theta - \sec^2 4^{-x}\theta.$ $\qquad s = \tan^2\theta$

Ex. 15. Let $s_x = \frac{\cos\frac{1}{3}\theta}{\sin\theta} + \frac{\cos\frac{1}{9}\theta}{\sin\frac{1}{3}\theta} + \ldots\ldots + \frac{\cos 3^{-x}\theta}{\sin 3^{-x+1}\theta}$.

$$u_{x+1} = \frac{\cos 3^{-x-1}\theta}{\sin 3^{-x}\theta},$$

and (eq. 45) $s_x = c + \frac{1}{2}\cot 3^{-x}\theta$

$= \frac{1}{2}(\cot 3^{-x}\theta - \cot\theta).$

Ex. 16. Let $s_x = \frac{\cos^3\frac{1}{3}\theta}{\sin\theta} + 3.\frac{\cos^3\frac{1}{9}\theta}{\sin\frac{1}{3}\theta} + \ldots\ldots + 3^{x-1}.\frac{\cos^3 3^{-x}\theta}{\sin 3^{-x+1}\theta}$.

$$u_{x+1} = 3^x.\frac{\cos^3 3^{-x-1}\theta}{\sin 3^{-x}\theta},$$

and (eq. 46) $s_x = c + \frac{1}{8}.3^x\cot 3^{-x}\theta$

$= \frac{1}{8}(3^x\cot 3^{-x}\theta - \cot\theta).$

Ex. 17. Let $s_x = \frac{\sin\frac{2}{3}\theta\sin\frac{1}{3}\theta}{\sin\theta} + \frac{1}{3}.\frac{\sin\frac{2}{9}\theta\sin\frac{1}{9}\theta}{\sin\frac{1}{3}\theta} + \ldots \frac{1}{3^{x-1}}.\frac{\sin 3^{-x}.2\theta\sin 3^{-x}\theta}{\sin 3^{-x+1}}.$

$$u_{x+1} = \frac{1}{3^x}\frac{\sin 3^{-x-1}.2\theta\sin 3^{-x-1}\theta}{\sin 3^{-x}\theta},$$

and (eq. 47) $s_x = c + \frac{3}{4}.3^{-x}\cot 3^{-x}\theta$

$= \frac{3}{4}(3^{-x}\cot 3^{-x}\theta - \cot\theta);$

$s = \frac{3}{4}\left(\frac{1}{\theta} - \cot\theta\right).$

Corollary. The series in *Ex.* 15 is evidently one fourth of the sum of the two series

$\cot\theta + 3\cot\frac{1}{3}\theta + 9\cot\frac{1}{9}\theta + \ldots\ldots 3^{x-1}.\cot 3^{-x+1}\theta,$

and $\frac{3\cos\frac{1}{3}\theta}{\sin\theta} + \frac{9\cos\frac{1}{9}\theta}{\sin\frac{1}{3}\theta} + \ldots\ldots\ldots 3^x.\frac{\cos 3^{-x}\theta}{\sin 3^{-x+1}\theta};$

and that in *Ex.* 16 is half the difference of the two

$$\frac{\cos\frac{1}{3}\theta}{\sin\theta}+\tfrac{1}{3}.\frac{\cos\frac{1}{9}\theta}{\sin\frac{1}{3}\theta}+\tfrac{1}{9}.\frac{\cos\frac{1}{27}\theta}{\sin\frac{1}{9}\theta}+\dots\dots\frac{1}{3^{x-1}}.\frac{\cos 3^{-x}\theta}{\sin 3^{-x+1}\theta},$$

and $\cot\theta+\frac{1}{3}.\cot\frac{1}{3}\theta+\frac{1}{9}\cot\frac{1}{9}\theta+\dots 3^{-x+1}.\cot 3^{-x+1}\theta.$

Examples of the products of Trigonometrical factors.

If

$$P_x = u_1 u_2 u_3 \dots\dots u_x,$$
$$P_{x+1} = u_1 u_2 u_3 \dots\dots u_{x+1};$$
$$\frac{P_{x+1}}{P_x} = u_{x+1},$$
$$\frac{P_{x+1}}{P_x} - 1 = \frac{P_{x+1}-P_x}{P_x} = \frac{\Delta\, P_x}{P_x} = u_{x+1}-1.$$

Hence, if we have, $\varphi(x)$ being any function of x,

$$u_{x+1}-1=\frac{\Delta.\,\varphi(x)}{\varphi(x)},$$

we shall have

$$P_x = c\,\varphi(x);$$
$$P_1 = u_1 = c\,\varphi(1).$$

But

$$u_{x+1} = \frac{\Delta.\,\varphi(x)}{\varphi(x)}+1$$
$$= \frac{\varphi(x+1)}{\varphi(x)},$$
$$u_1 = \frac{\varphi(1)}{\varphi(0)};$$

therefore

$$1 = c\,\varphi(0),$$
$$P_x = \frac{\varphi(x)}{\varphi(0)}.$$

Example 1. Let $\quad u_{x+1}-1 = \cos 2^x\theta - 1 = \dfrac{\Delta.\ 2^{-x}\sin 2^x\theta}{2^{-x}\sin 2^x\theta}$, (eq. 2)

$$u_{x+1} = \cos 2^x\theta,$$
$$u_x = \cos 2^{x-1}\theta;$$
$$\varphi(x) = 2^{-x}\sin 2^x\theta,$$
$$\varphi(0) = \sin\theta;$$

$$P_x = \cos\theta\cos 2\,\theta\cos 4\theta\dots\dots\cos 2^{x-1}\theta = \frac{\varphi(x)}{\varphi(0)} = \frac{1}{2^x}.\frac{\sin 2^x\theta}{\sin\theta}.$$

Corollary. Write $2^{-x+1}\theta$ for θ, and this becomes

$$P_x = \cos\theta\cos\tfrac{1}{2}\,\theta\,\cos\tfrac{1}{4}\,\theta\dots\dots\cos 2^{-x+1}\theta = \frac{1}{2^x},\frac{\sin 2\theta}{\sin 2^{-x+1}\theta},$$

and $\quad P_\infty = \cos\theta\,\cos\frac{1}{2}\,\theta\,\cos\frac{1}{4}\,\theta\dots\dots\dots = \dfrac{\sin 2\theta}{2\theta},$

which might also be deduced from equation 4.

Example 2. Let $\quad u_{x+1}-1 = 2\cos 3^x\theta = \dfrac{\Delta\sin 3^x.\,\frac{1}{2}\theta}{\sin 3^x.\,\frac{1}{2}\theta}$, (eq. 5)

$$u_x = 1+2\cos 3^{x-1}\,\theta$$
$$\varphi(x) = \sin 3^x.\,\tfrac{1}{2}\,\theta, \qquad \varphi(0) = \sin\tfrac{1}{2}\theta$$

$$P_x = (1+2\cos\theta)(1+2\cos 3\theta)(1+2\cos 9\theta)\ldots\ldots(1+2\cos 3^{x-1}\theta)$$
$$= \frac{\sin 3^x.\tfrac{1}{2}\theta}{\sin\tfrac{1}{2}\theta},$$

which is question (92) of the Miscellany.

Corollary. Write $3^{-x+1}\theta$ for θ, and this becomes

$$P_x = (1+2\cos\theta)(1+2\cos\tfrac{1}{3}\theta)(1+2\cos\tfrac{1}{9}\theta)\ldots\ldots(1+2\cos 3^{-x+1}\theta)$$
$$= \frac{\sin\tfrac{3}{2}\theta}{\sin 3^{-x+1}.\tfrac{1}{2}\theta}$$

Example 3. Let $u_{x+1} - 1 = -2\cos 3^x\theta = \dfrac{\Delta(-1)^x\cos 3^x.\tfrac{1}{2}\theta}{(-1)^x\cos 3^x.\tfrac{1}{2}\theta}$ (eq. 19)

$$u_x = 1 - 2\cos 3^{x-1}\theta,$$
$$\varphi(x) = (-1)^x\cos 3^x.\tfrac{1}{2}\theta, \qquad \varphi(0) = \cos\tfrac{1}{2}\theta;$$
$$P_x = (1-2\cos\theta)(1-2\cos 3\theta)\ldots\ldots(1-2\cos 3^{x-1}\theta)$$
$$= (-1)^x.\frac{\cos 3^x.\tfrac{1}{2}\theta}{\cos\tfrac{1}{2}\theta}.$$

Corollary. Write $3^{-x+1}\theta$ for θ, and this becomes

$$P_x = (1-2\cos\theta)(1-2\cos\tfrac{1}{3}\theta)\ldots\ldots\left(1-2\cos\frac{\theta}{3^{x-1}}\right)$$
$$= (-1)^x.\frac{\cos\tfrac{3}{2}\theta}{\cos 3^{-x+1}.\tfrac{1}{2}\theta},$$

Example 4. Let $u_{x+1} - 1 = \dfrac{2\cos 3^{-x-1}\theta}{\cos 3^{-x}\theta} = \dfrac{\Delta.\cot 3^{-x}\theta}{\cot 3^{-x}\theta}$, (eq. 45),

$$u_x = 1 + 2.\frac{\cos 3^{-x}\theta}{\cos 3^{-x+1}\theta},$$
$$\varphi(x) = \cot 3^{-x}\theta, \qquad \varphi(0) = \cot\theta$$
$$P_x = \left(1+2\frac{\cos\tfrac{1}{3}\theta}{\cos\theta}\right)\left(1+2\frac{\cos\tfrac{1}{9}\theta}{\cos\tfrac{1}{3}\theta}\right)\ldots\ldots\left(1+2\frac{\cos 3^{-x}\theta}{\cos 3^{-x+1}\theta}\right)$$
$$= \frac{\cot 3^{-x}\theta}{\cot\theta}.$$

Corollary. Write $3^x\theta$ for θ, and this becomes

$$P_x = \left(1+2\frac{\cos\theta}{\cos 3\theta}\right)\left(1+2\frac{\cos 3\theta}{\cos 9\theta}\right)\ldots\left(1+2\frac{\cos 3^{x-1}\theta}{\cos 3^x\theta}\right) = \frac{\cot\theta}{\cot 3^x\theta}$$

Example 5. Let $u_{x+1} - 1 = \dfrac{\cos 2^x.3\theta}{2\cos^3 2^x\theta} = \dfrac{\Delta.3^x\cot 2^x\theta}{3^x\cot 2^x\theta}$, (eq. 44);

$$u_x = 1+\tfrac{1}{2}.\frac{\cos 2^{x-1}.3\theta}{\cos^3 2^{x-1}\theta}$$
$$P_x = \left(1+\tfrac{1}{2}\frac{\cos 3\theta}{\cos^3\theta}\right)\left(1+\tfrac{1}{2}\frac{\cos 6\theta}{\cos^3 2\theta}\right)\ldots\ldots\left(1+\tfrac{1}{2}\frac{\cos 2^{x-1}}{\cos^3 2^{x-1}\theta}\right)$$
$$= 3^x.\frac{\cot 2^x\theta}{\cot\theta}$$

Corollary. Write $2^{-x+1}\theta$ for θ, and multiply by 2^x,

$$P_x = \left(2+\frac{\cos 3\theta}{\cos^3\theta}\right)\left(2+\frac{\cos\tfrac{3}{2}\theta}{\cos^3\tfrac{1}{2}\theta}\right)\ldots\ldots\left(2+\frac{\cos 2^{-x+1}.3\theta}{\cos^3 2^{-x+1}\theta}\right)$$
$$= \frac{6^x\cot 2\theta}{\cot 2^{-x+1}\theta}.$$

ARTICLE VII.

DIOPHANTINE SPECULATIONS.

By Wm. Lenhart, Esq. York, Pa.

NUMBER THREE.

In the Mathematical Diary, Vol. II, page 185, it is said, that, in No. 12, for June 1830, of the Annals of Mathematics pure and applied by M. Gergonne, M. Pagliani resolves this problem: " To find 1000 consecutive numbers of the natural series, such that the sum of their cubes shall be itself a cube," and also that M. Pagliani thinks that no other solution can be deduced, except by a very complicated analysis. We have not had the pleasure of seeing the number alluded to, nor, indeed, any of the numbers of M. Gergonne's celebrated work, and are therefore not aware of the method M. P. has pursued to obtain his numbers. As we deem the Problem, when generalized, very beautiful, we propose laying it before the readers of the Miscellany, accompanied by an ample solution, entirely free from a complicated analysis; and for which we claim their indulgence.

Problem. To find m numbers in arithmetical progression such that the sum of their cubes shall be itself a cube; and give examples when the numbers are consecutive numbers of the natural series.

SOLUTION.

Case I. When m is of the form $2n$, or an even number.

Let us assume the equation $x^3+y^3=(x+y)\{x(x-y)+y^2\}$. (a)

and suppose $\left\{\begin{matrix} x=s+1, & s+2, & s+3, & \&c \\ y=s & s-1, & s-2, & \&c. \end{matrix}\right\}$ respectively.

Then will $x+y=2s+1$, and $x-y=1, 3, 5, 7$, &c.; and by substituting the different values of x and y in equation (a) we shall find

$$(s+1)^3+(s)^3=(2s+1).(s^2+s+1) \quad \ldots \quad (1),$$
$$(s+2)^3+(s-1)^3=(2s+1).(s^2+s+7) \quad \ldots \quad (2),$$
$$(s+3)^3+(s-2)^3=(2s+1).(s^2+s+19) \quad \ldots \quad (3),$$
$$\&c. \qquad \&c.$$

Now the sum of ..... (1). (2) or of 4 cubes is $(2s+1).(2s^2+2s+8)$,
of .... (1). (2). (3) " 6 " $(2s+1).(3s^2+3s+27)$,
of .. (1).(2).(3) to (n)" $2n$ " $(2s+1).(ns^2+ns+n^3)$;

Or making the multiplication

$$2ns^3+3ns^2+(2n^3+n)s+n^3 \quad . \quad . \quad . \quad . \quad (b)$$

which is a general expression for the sum of an even number of cubes, and which, by the problem, is to be made a cube. Put $n=4n'^3$, and divide by $(2n')^3$ then

$$s^3+\frac{3s^2}{2}+\left(\frac{32n'^6+1}{2}\right)s+8n'^6=\text{cube}=(s+2n'^2)^3:$$

This reduced will give

$$s=\frac{32n'^6-24n'^4+1}{12n'^2-3}=\frac{8n'^4-4n'^2-1}{3}.$$

Now, in order that the above series of values of x and y may be consecutive numbers of the natural series, it is evident that s must be a whole number greater than unity, which, as may be easily proved, will always be the case when n' is prime to or not divisible by 3.*

Let us suppose $n'=2$, then $n=32$, $m=64$, $s=37$ and $s+1=38$; and as 37 and 38 are the means of the 64 numbers we shall of course have 6 and 69 for the extremes; that is, 6 will be the first, and 69 the last of 64 consecutive numbers of the natural series such that the sum of their cubes shall be itself a cube: and its root $2n'\,(s+2n'^2)=180$.

Suppose $n'=5$, then $n=500$, $m=1000$ and $s=1633$, $s+1=1634$. Consequently 1134 will be the first, and 2133 the last of 1000 consecutive numbers of the natural series having the same properties: and $2n'\,(s+2n'\ )=16830$ will be the root of the cube to which the sum of their cubes is equal. These numbers are the same as those named in the Diary, as having been found by M. Pagliani.

But let us return to the original formula

$$2ns^3+3ns^2+(2n^3+n)\,s+n^3,$$

and make a cube of it by assuming the root $n+\left(\dfrac{2n^2+1}{3n}\right)s$.

From this assumption by reduction, we find

$$s=\frac{9n^2\,(-4n^4+5n^2-1)}{8n^6-42n^4+6n^2+1}=\frac{9n^2\,(n+1).(n-1)}{2n^2\,(5-n^2)+1}$$

in which n may be taken for any number $>$ than unity. If $n=2$, then $m=4$, $s=12$, $s+1=13$, and the four roots will be the consecutive numbers 11, 12, 13 and 14. If $n=3$, then $m=6$; $s=-\frac{648}{71}$, $s+1=-\frac{577}{71}$ and thence the progression 435, 506, 577, 648, 719, 790, having a common difference 71, and the sum of their cubes $(1155)^3$.

Case II. When m is of the form $2n+1$, or an odd number.

By adding $(s-n)^3$ to formula (b) we shall find

$$(2n+1)\,s^3+(2n+1).(n+1)\,ns \quad . \quad . \quad . \quad (c),$$

which is evidently a general expression for the sum of an odd number of cubes. It may be simplified by putting m, which, in this case, is an odd number in the place of $2n+1$. Formula (c) then becomes

$$ms^3+\left(\frac{m\,(m^2-1).}{4}\right)s \quad . \quad . \quad . \quad . \quad (d),$$

which by the problem, is to be made a cube. It is a cube when $s=\frac{1}{2}$; put therefore $s=\frac{1}{2}+t$, and (d) becomes

$$\frac{m^3}{8}+\left(\frac{m^3+2m}{4}\right)t+\frac{3m}{2}\,t^2+mt^3 \quad . \quad . \quad (f),$$

or, putting $m=m'^3$, and dividing by m'^3

$$\frac{m'^6}{8}+\left(\frac{m'^6+2}{4}\right)t+\frac{3t^2}{2}+t^3=\text{cube}=\left(\frac{m'^2}{2}+t\right)^3.$$

---

* If, however, we take $n'=1$, we shall have $n=4$, $m=8$, $s=1$, $s+1=2$, and the eight roots will be 2, 1, 3, 0, 4, -1, 5 and -2; but four of these cancel each other, and one of them is 0, there will therefore only remain the consecutive roots 3, 4 and 5. And thus we curiously obtain the well known equation $(3)^3+(4)^3+(5)^3=(6)^3$.

This being reduced will furnish

$$t = \frac{m'^6 - 3m'^4 + 2}{6\,(m'^2 - 1)} = \frac{m'^4 - 2m'^2 - 2}{6} \text{ and therefore}$$

$$s = \tfrac{1}{2} + t = \frac{(m'^2 - 1)^2}{6},$$

which will evidently always be a whole number when m' is an odd number $>$ than unity, and prime to, or not divisible by 3.* Assume $m' = 5$ then will $m = 125$, $s = 96$, $s + 1 = 97$, $n = 62$, $s - n = 34$, $s + n = 158$; and therefore 34 will be the first, and 158 the last, of 125 consecutive numbers of the natural series such that the sum of their cubes shall be itself a cube; and $m'\left(\frac{m'^2}{2} + t\right) = 540$ will be its root. If we take $m' = 9$ we shall find 2108 to be the first, and 4292 the last of 729 numbers in arithmetical progression, having 3 for a common difference, and the sum of their cubes $(9960)^3$.

Again: equate (f) to $\left(\frac{m}{2} + \left(\frac{m^2 + 2}{3m}\right) t\right)^3$, and we shall find $s = \frac{4\,(m^2 - 1)^2}{9\,(2m^2 + 1) - (m^2 - 1)^2}$, in which any odd number $>$ than unity may be taken for m, and thence integers obtained to answer.

There is another, and, perhaps, a preferable method of making formula (d) a cube. It is briefly this: put $ms^3 + \left(\frac{m\,(m^2 - 1)}{4}\right)s = p^3 m^3 s^3$, then, by dividing by ms, and reducing, &c., we shall find

$$4s^2 = \frac{m^2 - 1}{p^3 m^2 - 1} = \square \quad . \quad . \quad . \quad (g).$$

Now (g) is plainly a square when $p = 1$; put then $p = 1 + r$, and from the reciprocal of (g) we shall have

$$1 + \frac{3m^2}{m^2 - 1} r + \frac{3m^2}{m^2 - 1} r^2 + \frac{m^2}{m^2 - 1} r^3 = \square = \left(1 + \frac{3m^2}{2\,(m^2 - 1)} r\right)^2.$$

$$r = \frac{3\,(4 - m^2)}{4\,(m^2 - 1)};\ \text{then } p = 1 + r = \frac{m^2 + 8}{4\,(m^2 - 1)} \text{ and consequently}$$

$$s = \frac{4\,(m^2 - 1)^2}{9\,(2\,m^2 + 1) - (m^2 - 1)^2}, \text{ the same as before.}$$

Equation (g) is also a square when $m = 3$ and $p = \frac{1}{2}$. Then $s = 4$, and we obtain, as before, the three consecutive numbers 3, 4 and 5.

Problem. Find three square numbers such that the difference of every two of them may be squares.

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* We shall take this occasion to note that Barlow in his miscellaneous propositions, page 258 of his Theory of Numbers, has the following proposition, namely: "If m be a prime number greater than 3, then will $m^2 - 1$ be divisible by 24." Now this is certainly true; but it is also true if m be any odd number not divisible by 3, which, if thus enunciated, would make the proposition more general.

Solution. If we suppose the roots of the three required squares to be represented by $\frac{x^2+y^2}{x^2-y^2}$, $\frac{v^2+w^2}{v^2-w^2}$ and unity; or their reciprocals, unity, $\frac{v^2-w^2}{v^2+w^2}$ and $\frac{x^2-y^2}{x^2+y^2}$. Or $\frac{v^2+w^2}{2vw}$, $\frac{x^2+y^2}{2xy}$ and unity; or their reciprocals, unity, $\frac{2xy}{x^2+y^2}$, and $\frac{2vw}{v^2+w^2}$; then will the following conditions be answered, namely:

$$\left(\frac{x^2+y^2}{x^2-y^2}\right)^2-1=\left(\frac{2xy}{x^2-y^2}\right)^2;\ \left(\frac{v^2+w^2}{v^2-w^2}\right)^2-1=\left(\frac{2vw}{v^2-w^2}\right)^2;$$

$$1-\left(\frac{v^2-w^2}{v^2+w^2}\right)^2=\left(\frac{2vw}{v^2+w^2}\right)^2;\ 1-\left(\frac{x^2-y^2}{x^2+y^2}\right)^2=\left(\frac{2xy}{x^2+y^2}\right)^2;$$

$$\left(\frac{v^2+w^2}{2vw}\right)^2-1=\left(\frac{v^2-w^2}{2vw}\right)^2;\ \left(\frac{x^2+y^2}{2xy}\right)^2-1=\left(\frac{x^2-y^2}{2xy}\right)^2;$$

$$1-\left(\frac{2xy}{x^2+y^2}\right)^2=\left(\frac{x^2-y^2}{x^2+y^2}\right)^2 \text{ and } 1-\left(\frac{2vw}{v^2+w^2}\right)^2=\left(\frac{v^2-w^2}{v^2+w^2}\right)^2:$$

and it will therefore only remain to make squares of the formulas following, viz.:

$$\text{I.}\qquad \left(\frac{x^2+y^2}{x^2-y^2}\right)^2-\left(\frac{v^2+w^2}{v^2-w^2}\right)^2$$

$$\text{II.}\qquad \left(\frac{v^2-w^2}{v^2+w^2}\right)^2-\left(\frac{x^2-y^2}{x^2+y^2}\right)^2$$

$$\text{III.}\qquad \left(\frac{v^2+w^2}{2vw}\right)^2-\left(\frac{x^2+y^2}{2xy}\right)^2$$

$$\text{IV.}\qquad \left(\frac{2xy}{x^2+y^2}\right)^2-\left(\frac{2vw}{v^2+w^2}\right)^2$$

Now, if I, II, III and IV, be properly reduced, and the square factors and denominators be rejected, it will be found that each will require the same formula to be made a square, namely:

$$(vx+wy).(vx-wy).(vy+wx).(vy-wx) \quad . \quad . \quad . \quad (5)$$

and therefore the same values of v, w, x and y, that shall render (5) a square, will also enable us at once to obtain four sets of integers to answer, which is exceedingly curious; and, as we presume, something entirely new on this old and often discussed subject.

To make (5) a square, we shall proceed thus: put

$$vy+wx=t\,(vx-wy) \quad . \quad . \quad . \quad . \quad (6)$$

then substituting in (5) and dividing by $(vx-wy)^2$ we shall have to make

$$t\,(vx+wy).(vy-wx)=\square \quad . \quad . \quad . \quad (7).$$

From (6) we obtain $\frac{v}{w}=\frac{ty+x}{tx-y}$, therefore $v=ty+x$ and $w=tx-y$

$$t\,(x^2-y^2+2txy).(ty^2-tx^2+2xy)=\square, \text{ Consequently}$$

$$x^2-y^2+2txy=\square \quad . \quad . \quad . \quad . \quad . \quad . \quad (8),$$

$$y^2-x^2+\frac{2xy}{t}=\square \quad . \quad . \quad . \quad . \quad (9).$$

These formulas are identical with those in the Ladies' Diary, Vol. IV, page 346, and in the American edition of Young's Algebra, page 350, to which we might therefore refer, but as they are there rather laboriously and not neatly reduced, we have concluded to resolve them here at length.

In the first place, then (9) is a square, when $x=\frac{2y}{t}$; consequently (8) becomes $4+3t^2=\square=(2-pt)^2$; from which $t=\frac{4p}{p^2-3}$, therefore $x=p^2-3$ and $y=2p$. Then $v=(p^2+1)^2+8$ and $w=2p\,(p^2-3)$. If $p=2$, then $x=1$, $y=4$, $v=33$, $w=4$ and the required roots will be

$\frac{17}{15}$, $\frac{1105}{1073}$ and 1; $\frac{15}{17}$, $\frac{1073}{1105}$ and 1; $\frac{1105}{8\times 33}$, $\frac{17}{8}$ and 1; $\frac{8\times 33}{1105}$, $\frac{8}{17}$ and 1; Or, reduced to integers

16095, 16575 and 18241; 264, 561 and 1105;
975, 1073 " 1105; 264, 520 " 1105.

In the second place, put formula (8) or

$$x^2-y^2+2txy=\square=(x-py)^2;$$

then $x=p^2+1$, aud $y=2\,(p+t)$: assume also

$$t^2y^2-t^2x^2+2t\,xy=\square=(ty-r)^2$$

then $-t^2x^2+2\,txy=-2\,rty+r^2$; or substituting for y its value $2\,(p+t)$; $-t^2x^2+4ptx+4t^2x=-4pxt-4rt^2+r^2$.

Now, put $-t^2x^2+4t^2x=-4rt^2$ and $4ptx=-4prt+r^2$, then

$$\text{will } r=\frac{x^2-4x}{4} \text{ and } t=\frac{r^2}{4\,p\,(r+x)}=\frac{r^2}{px^2}=\frac{(p^2-3)^2}{16p}.$$

Consequently $x=p^2+1, y=2\,(p+t)=\frac{(p^2-3)^2+16p^2}{8p}=\frac{(p^2+1).(p^2+9)}{8p}$

or dividing by $\frac{p^2+1}{8p}$, $x=8p$ and $y=p^2+9$.

Again: $v=ty+x=\frac{(p^2-3)^2.(p^2+9)+128p^2}{16p}=\frac{(p^2+1).(p^4+2p^2+81)}{16p}$

and $w=tx-y=\frac{(p^2-3)^2}{2}-p^2-9=\frac{p^4-8p^2-9}{2}=\frac{(p^2+1).(p^2-9)}{2}$

or, dividing by $\frac{p^2+1}{16p}$ we shall get $v=p^4+2p^2+81=(p^2+4p+9).(p^2-4p+9)$, and $w=8p\,(p^2-9)$.

Now, if $p=1$, then $x=4$, $y=5$, $v=21$, $w=16$ and thence the roots, $\frac{41}{9}$, $\frac{697}{185}$ and 1; $\frac{9}{41}$, $\frac{185}{697}$ and 1; $\frac{697}{672}$, $\frac{41}{40}$ and 1; $\frac{672}{697}$, $\frac{40}{41}$ and 1: Or, in integers

1665, 6273 and 7585; 3360, 3444 and 3485;
672, 680 and 697; 153, 185 and 697.

We may also make (7) a square in the following manner:

From (6) we get $\frac{x}{y}=\frac{tw+v}{tv-w}$, or $x=tw+v$ and $y=tv-w$.

Hence (7) becomes $t(v^2-w^2+2tw).(tv^2-tw^2-2vw)=\square$,
and therefore $v^2-w^2+2tvw=\square=A^2$ . . . . . (10)

$$v^2-w^2-\frac{2vw}{t}=\square=B^2 \quad . \quad . \quad . \quad . \quad . \quad (11)$$

Now $(10)-(11)=(A+B).(A-B)=2v\times\frac{w(t^2+1)}{t}$, and putting $A+B=2v$ and $A-B=\frac{w(t^2+1)}{t}$ we shall find $A=v+\frac{w(t^2+1)}{2t}$ and thence from (10), $v=(t^2+1)^2+4t^2$ & $w=4t(t^2-1)$; also $x=tw+v=(t^2+1)^2+4t^4$ and $y=tv-w=t((t^2+1)^2+4)$, in which $t>1$. If $t=3$, then $v=17$, $w=12$, $x=53$, $y=39$, and the two least sets of roots will be found to be 644, 725 and 2165; and 2040, 2067 and 2165.

Problem Find three squares such that the sum of every two shall make a square.

Solution. Suppose $\left(\frac{p^2-1}{2p}\right)^2$, $\left(\frac{2q}{q^2-1}\right)^2$ and unity, to be the three required squares, then will

$\left(\frac{p^2-1}{2p}\right)^2+1=\left(\frac{p^2+1}{2p}\right)^2=\square$, and $\left(\frac{2q}{q^2-1}\right)^2+1=\left(\frac{q^2+1}{q^2-1}\right)^2=\square$;

and therefore it only remains to make

$$\left(\frac{p^2-1}{2p}\right)^2+\left(\frac{2q}{q^2-1}\right)^2=\square;$$

Or, actually performing the addition, and rejecting the square denominator, to make

$(p^2-1)^2.(q^2-1)^2+16p^2q^2=\square=((p^2-1).(q^2-1)+8)^2$ by assumption.

This, by a proper reduction, furnishes the extraordinary equation

$$p^2+q^2=5=(1)^2+(2)^2,$$

which is resolved in all the elementary books, and furnishes

$$p=\frac{s^2+4s-1}{s^2+1} \text{ and } q=\frac{2(s^2-s-1)}{s^2+1}$$

in which s may be any number other than unity or three. If $s=2$, then

$$p=\frac{11}{5},\ q=\frac{2}{5} \text{ and the required squares will be}$$

$\left(\frac{48}{55}\right)^2$, $\left(\frac{20}{21}\right)^2$ and unity; or their reciprocals $\left(\frac{55}{48}\right)^2$, $\left(\frac{21}{20}\right)^2$ and 1.

Or integers

$(1008)^2$, $(1100)^2$ and $(1155)^2$; $(240)^2$, $(252)^2$ and $(275)^2$.

WM. LENHART.

YORK, Penn., *January*, 1839.

ARTICLE VIII.

ON THE APPLICATION OF STURM'S THEOREM.

I. *To Equations of the Fourth Degree.*

Any equation of the fourth degree may be put into the form

$$\mathrm{X} = x^4 + ax^2 + bx + c = 0;$$

and applying the process of Sturm, we find

$$\mathrm{X}_1 = 4x^3 + 2ax + b,$$
$$\mathrm{X}_2 = -2ax^2 - 3bx - 4c,$$
$$\mathrm{X}_3 = (8ac - 2a^3 - 9b^2)\, x - b\,(a^2 + 12c)$$
$$= \mathrm{A}x + \mathrm{B},$$
$$\mathrm{X}_4 = 4c\mathrm{A}^2 - 3b\mathrm{AB} + 2a\mathrm{B}^2$$
$$= 4\,(a^2 + 12c)^3 - (27b^2 + 2a^3 - 72ac)^2.$$

Making, for the co-efficients of the polynomial X_3,

$$\mathrm{A} = 8ac - 2a^3 - 9b^2,$$
$$\mathrm{B} = -b\,(a^2 + 12c).$$

In calculating these co-efficients, if those of any one polynomial, as, for instance, A and B, have a common divisor, that divisor may be neglected, if it is a numerical quantity, or any quantity essentially positive. In cases also where the ordinary methods of approximation might be advantageously employed, that is, when the roots are widely separated, approximate values of these co-efficients may be used which are nearly proportional to them. If any of the leading or terminating co-efficients of the succeeding polynomials are found to approximate to zero, the true values must be recurred to; for under such circumstances, there may be roots either equal, or having a very small difference.

Now for $x = \infty$, these polynomials will become

$$\mathrm{X} = +\infty, \quad \mathrm{X}_1 = +\infty, \quad \mathrm{X}_2 = -a.\infty, \quad \mathrm{X}_3 = \mathrm{A}.\infty, \quad \mathrm{X}_4 = \text{const.}$$

Let i be the number of changes of sign in these quantities, and represent by m the number of real roots of the equation $\mathrm{X} = 0$; then, as a necessary consequence of Sturm's Theorem, when the number of polynomials is complete,

$$m = \pm\,(4 - 2\,i) = \pm\,2\,(2 - i).$$

Thus we may form the following Table of conditions: when

		$\mathrm{X}_4 > 0$		$\mathrm{X}_4 < 0$	
$a > 0$	$\mathrm{A} > 0$	$i = 2$	$m = 0$	$i = 3$	$m = 2,$
"	$\mathrm{A} < 0$	2	0	1	2,
$a < 0$	$\mathrm{A} > 0$	0	4	1	2,
"	$\mathrm{A} < 0$	2	0	1	2,

So that, if $\mathrm{X}_4 < 0$, or if

$$(27b^2 + 2a^3 - 7ac)^2 > 4\,(a^2 + 12c)^3,$$

the equation has two real roots and no more; while if $\mathrm{X}_4 > 0$, or

$$(27b^2 + 2a^3 - 7ac)^2 < 4\,(a^2 + 12c)^3,$$

the equation has either four real roots, or none at all, and it has four if, at the same time,

$$a < 0 \text{ and } \mathrm{A} = 8\,ac - 2\,a^3 - 9b^2 > 0,$$

It may be remarked that we cannot have $i = 3$; for X_4 may be put into the form

$$X_4 = \tfrac{1}{a}\{64\, a^6 b^2 + 2a^3\,(216\, b^4 + 36b^2\, A + A^2) + (9b^2 + A)^3\};$$

so that if $a > 0$ and $A > 0$,

we necessarily have $X_4 > 0$.

To find the nature of these roots, let $x = 0$, and the polynomials are

$$X = c, \quad X_1 = b, \quad X_2 = -c, \quad X_3 = B, \quad X_4 = \text{const}.$$

Let k be the number of changes of sign in these quantities, which is obviously not affected by any change in the value of b, otherwise than as this change may affect the sign of B, and m' the number of *positive* roots in the equation $X = 0$. Then will

$$m' = \pm\,(i - k);$$

and the only cases we have to examine are, 1°. when there are four real roots, or $X_4 > 0$ and $i = 0$, and, 2°. when there are two real roots or $X_4 < 0$ and $i = 1$. Then when

		$X_4 > 0,\ i = 0$		$X_4 < 0$ and $i = 1$,	
$c > 0$	$B > 0$	$k = 2$	$m' = 2$	$k = 3$	$m' = 2$,
"	$B < 0$	$= 2$	2	1	0,
$c < 0$	$B > 0$	$= 1$	1	2	1,
"	$B < 0$	$= 3$	3	2	1,

so that, if all the four roots are real,

2 are positive and 2 negative, if $c > 0$;

1 is positive and 3 negative, if $c < 0$ and $B > 0$ or $b\,(a^2 + 12c) < 0$;

3 are positive and one negative, if $c < 0$ and $B < 0$ or $b\,(a^2 + 12c) > 0$.

If only two of the roots are real,

1 of them is positive and 1 negative, if $c < 0$;

both are positive, if $c > 0$ and $B > 0$ or $b\,(a^2 + 12c) < 0$;

both are negative, if $c > 0$ and $B < 0$ or $b\,(a^2 + 12c) > 0$.

It is evident that if $X_4 = 0$, or

$$(27b^2 + 2a^3 - 7ac)^2 = 4\,(a^2 + 12c)^3,$$

two of the roots are equal, and these are each

$$= -\frac{B}{A} = \frac{b\,(a^2 + 12c)}{8\,ac - 2a^3 - 9b^2},$$

and the other roots are those of the equation

$$(Ax - B)^2 + A^2 a + 2B^2 = 0.$$

If $A = 0$ and $B = 0$, there are two pairs of equal roots, which are equal to the roots of the equation

$$2ax^2 + 3bx + 4c = 0;$$

except when the roots of this last equation are equal, that is when

$$9b^2 - 32\, ac = 0,$$

and then there are three roots each $= -\dfrac{3b}{4a}$, the fourth one being $\dfrac{9b}{4a}$.

If, $A = 0$, while B is finite, $X_3 = B$, and the Theorem gives in this case two real roots when a and B have contrary signs, and no real roots otherwise; these two roots are

both positive if B and c are both positive;

both negative if B and c are both negative;

one positive and one negative if B and c have different signs.

If $a = 0$, or the equation is

$$X = x^4 + bx + c = 0,$$

the previous conclusions are no longer applicable; for then

$$X_1 = 4x^3 + b,$$
$$X_2 = -3bx - 4c,$$
$$X_3 = b\,(256c^3 - 27b^4);$$

and it is easily seen that the equation can never have more than two real roots, and two only when b and X_3 have different signs, that is when

$$256c^3 - 27.\,b^4 < 0;$$

also, these two roots have different signs if $c < 0$;

they are both positive if $c > 0,\ b < 0$;

they are both negative if $c > 0,\ b > 0$;

and they are each $= -\dfrac{4c}{3b}$, if $256c^3 - 27\,b^4 = 0$.

EXAMPLE.

Take the equation (8) in the solution of question (107),

$$2t^4 - kt^3 + 3t^2 - 2kt + 1 = 0,$$

and put $\qquad k = 2h, \qquad t = \dfrac{1}{x+h}, \quad$ then

$$X = x^4 + 3\,(1-2h^2)\,x^2 + 4\,(1-2h^2)\,hx + (1-h^2)\,(2+3h^2) = 0;$$
$$a = 3\,(1-2h^2), \qquad b = 4h\,(1-2h^2), \qquad c = (1-h^2)\,(2+3h^2);$$
$$A = (1-2h^2)(16h^2-1), \qquad B = 2h\,(1-2h^2)(8h^2-11);$$
$$X_4 = -\,128\,h^6 - 795\,h^4 + 597\,h^2 + 2.$$

The equation $X_4 = 0$, will be found, by the Theorem, to have three real values of h^2, two negative and one positive; the latter one, which is the only one that is here admissible, is

$$h^2 = ,680158;$$

so that, $\quad$ if $h^2 < ,680158$, or $k^2 < 2,720632$; X_4 is > 0.

But then, $\quad$ if $h^2 > \frac{1}{2}$; $\quad a < 0, \quad A < 0$;

if $h^2 < \frac{1}{2} > \frac{1}{16}$; $\quad a > 0, \quad A < 0$;

if $h^2 < \frac{1}{16}$; $\quad a > 0, \quad A > 0$,

so that the equation has no real roots.

If $h^2 = ,680158$ or $k^2 = 2.720632$; $X_4 = 0$

and the equation has a pair of real roots, each $= ,463888$.

If $h^2 > ,680158$ or $k^2 > 2,720632$; $X_4 < 0$,

and the equation has two real roots; also

if $h < 1$, or $k < 2$; $\quad c > 0, \quad B > 0$,

and both roots are positive;

if $h = 1$, or $k = 2$; $c = 0$,

and one root is zero, then $t = \dfrac{1}{h} = 1$;

if $h > 1$, or $k > 2$; $c < 0$.

and the roots have different signs.

The negative value of x is always $> -h$; for since $c < 0, X_4 < 0$, the polynomials have two changes of sign by the Table, when $x = 0$; and when $x = -h$, h varying from 1 to ∞, the polynomials are

$X=2>0$, $X_1=-h<0$, $X_2=h^2-2$, $X_3=2h^3-h>0$, $X_4<0$; so that there are three changes of sign, and consequently one real root between 0 and $-h$ while h is >1; it follows that both the resulting values of t are positive.

2. *To equations of the fifth Degree.*

Every equation of the fifth degree may be put into the form

$$X = x^5+ax^3+bx^2+cx+d=0,$$

and for this equation, we find

$$X_1 = 5x^4+3ax^2+2bx+c,$$
$$X_2 = -2ax^3-3bx^2-4cx-5d.$$
$$X_3 = Ax^2+Bx+C,$$
$$X_4 = Dx+E,$$
$$X_5 = BDE-CD^2-AE^2;$$

making thus, for the co-efficients of the polynomial X_3,

$$A = 40ac-12a^3-45b^2,$$
$$B = 50ad-8a^2b-60bc,$$
$$C = -4a^2c-75bd;$$

and, for those of the polynomial X_4,

$$D = 4cA^2+2aB^2-2aAC-3bAB,$$
$$E = 5dA^2+2aBC-3bAC;$$

the same remarks apply to the calculation of these co-efficients, as to those of the fourth degree. If $x=\infty$,

$X=+\infty$, $X_1=+\infty$, $X_2=-a.\infty$, $X_3=A.\infty$, $X_4=D.\infty$, $X_5=$const.; and using i and m as before, having in this case

$$m=\pm(5-2i),$$

the table of conditions is

			$X_5>0$		$X_5<0$	
$a>0$	$A>0$	$D>0$	$i=2$	$m=1$	$i=3$	$m=1$
"	"	$D<0$	4	3	3	1
"	$A<0$	$D>0$	2	1	3	1,
"	"	$D<0$	2	1	1	3,
$a<0$	$A>0$	$D>0$	0	5	1	3,
"	"	$D<0$	2	1	1	3,
"	$A<0$	$D>0$	2	1	3	1,
"	"	$D<0$	2	1	1	3.

The second of this series of conditions is impossible; since,

if $a>0$, $A>0$;

$40ac=A+12a^3+45b^2>0$, and $c>0$;

then $D=4cA^2+2aB^2-A(2ac+3bB)$

$= 4cA^2+2aB^2+4cA(2a^3+45b^2)+24a^2b^2A$

$>0.$

It follows that, if $X_5>0$, there must be either five real roots, or only one; and there are five, only when, at the same time,

$a<0.$ $A>0.$ $D>0.$

If $X_5 < 0$, there may be one or three real roots, but no more; and there will be three when, at the same time, either

$$1^\circ, \quad A < 0, \quad D < 0,$$
$$\text{or } 2^\circ, \quad a < 0, \quad A > 0.$$

To find the nature of these roots: if $x=0$, the polynomials become

$$X=d, \quad X_1=c, \quad X_2=-d, \quad X_3=C, \quad X_4=E, \quad X_5=\text{const.}^*$$

Now, when there is only one real root, its sign, from simple considerations, is opposite to that of d; and therefore we need only examine the two cases of five real roots, indicated by $X_5 > 0$ and $i=0$, and three real roots, indicated by $X_5 < 0$ and $i=1$. Then using k and m' as before, so that $m' = \pm(i-k)$, the Table of conditions is

			$X_5>0$ and $i=0$		$X_5<0$ and $i=1$	
$d>0$	$C>0$	$E>0$	$k=2$	$m'=2$	$k=3$	$m'=2$,
"	"	$E<0$	4	4	3	2,
"	$C<0$	$E>0$	2	2	3	2,
"	"	$E<0$	2	2	1	0,
$d<0$	$C>0$	$E>0$	1	1	2	1,
"	"	$E<0$	3	3	2	1,
"	$C<0$	$E>0$	3	3	4	3,
"	"	$E<0$	3	3	2	1,

Then when there are five real roots,
if $d>0$, 4 are positive and 1 negative, when $C>0$, $E<0$;
2 are positive and 3 negative, in all other cases:
if $d<0$, 1 is positive and 4 negative, when $C>0$, $E>0$;
3 are positive and 2 negative, in all other cases.
Also, when there are three real roots,
if $d>0$, all of them are negative, when $C<0$, $E<0$;
2 are positive and 1 negative, in all other cases.
if $d<0$, all of them are positive, when $C<0$, $E>0$;
1 is positive and 2 negative, in all other cases.

If $X_5 = 0$, there are a pair of equal roots, each $= -\frac{E}{D}$, and the other three are the roots of the equation

$$D^3x^3-2D^2Ex^2+(aD^2+3E^2)\,Dx+BD^3-2aD^2E-4E^3=0.$$

If $D=0$ and $E=0$, there are two pairs of equal roots, which are the roots of the equation

$$Ax^2+Bx+C=0,$$

the fifth root being $= -\frac{2B}{A}$; except when the roots of this equation are equal, or when

$$B^2-4AC=0,$$

and then there are three roots, each $= -\frac{B}{2A}$, and the other two are those of the equation

$$2\,AB^2x^2-3B^4x+16A^4d=0$$

If $A = 0$, $B = 0$ and $C = 0$, which could only be when, either

$$a = 10\, a'^2, \quad b = \pm 20a'^3, \quad c = -15a'^4, \quad d = \pm 4a'^5,$$

and then 4 roots are each $= \pm\, a'$ and the fifth $= \mp\, 4a'$; or

$$a = -15a'^2, \quad b = \pm\, 10a'^3, \quad c = 60a'^4, \quad d = \mp 72a'^5,$$

and then 3 roots are each $= \pm 2a'$ and 2 are each $= \mp 3a'$.

If $D=0$, $X_4 = E$, and the general results do not apply; it will be found that the equation has then three real roots,

$$\text{if } a < 0, \quad A > 0;$$

and that, of these roots,

if $d>0$, all of them are negative when $C < 0$, $E < 0$;
two positive and one negative in other cases:
if $d<0$, all of them are positive when $C < 0$, $E > 0$;
one positive and two negative in other cases;

otherwise, the equation has but one real root which has the opposite sign to d.

If $A=0$. then
$$X_3 = Bx + C,$$
$$X_4 = B\,(3bBC^2 + 5dB^2 - 2aC^2 - 4cB^2C);$$

and the equation has three real roots

$$\text{if } a<0, \quad B>0;$$

these roots being,

if $d>0$, all negative when $C<0$, $X_4<0$,
2 positive and 1 negative in other cases;
if $d<0$, all positive when $C < 0$, $X_4 > 0$,
1 positive and two negative in other cases:

otherwise, the equation has but one real root which has the opposite sign to d.

If $A=0$ and $B=0$; then $X_3=C$, and the equation has three real roots

$$\text{if } a<0, \quad C>0;$$

of which, if $d>0$, two are positive,
if $d<0$, one is positive,

otherwise it has only one real root.

If $a=0$, the equation and polynomials are

$$\begin{aligned}
X &= x^5 + bx^2 + cx + d = 0,\\
X_1 &= 5x^4 + 2bx + c,\\
X_2 &= -3bx^2 - 4cx - 5d,\\
X_3 &= Ax + B,\\
X_4 &= 5dA^2 - 4cAB + 3bB^2;
\end{aligned}$$

where
$$\begin{aligned}
A &= b\,(320c^3 - 600bcd - 54b^4),\\
B &= b\,(400c\ d - 375bd^2 - 27b^3c).
\end{aligned}$$

It is found that this equation has three real roots, if A and X_4 have the same sign, provided b have a contrary sign; and that of these roots,

if $d>0$, all are negative when $B<0$, $X_4<0$,
two positive in other cases;
if $d<0$, all are positive if $B<0$. $X^4>0$,
1 is positive and 2 negative in other cases;

otherwise there is but one real root, which has the contrary sign to d.

If $a=0$ and $b=0$, the equation and the polynomials are

$$\begin{aligned}
\mathrm{X} &= x^5+cx+d=0,\\
\mathrm{X}_1 &= 5x^4+c,\\
\mathrm{X}_2 &= -4cx-5d,\\
\mathrm{X}_3 &= -(\tfrac{1}{5}c)^5-(\tfrac{1}{4}d)^4.
\end{aligned}$$

and this equation has three real roots when

$$\mathrm{X}_3>0, \text{ or } -(\tfrac{1}{5}c)^5>(\tfrac{1}{4}d)^4,$$

of which only one is positive when $d<0$, and two are positive when $d>0$; otherwise it has only one real root.

As it may often spare the labor of a transformation, we shall also give, while on the subject, the application of the Theorem

3. *To equations of the Third Degree,*

in their most general form, that is when

$$\begin{aligned}
\mathrm{X} &= x^3+ax^2+bx+c=0,\\
\mathrm{X}_1 &= 3x^2+2ax+b,\\
\mathrm{X}_2 &= 2(a^2-3b)x+ab-9c\\
&= \mathrm{A}x+\mathrm{B},\\
\mathrm{X}_3 &= -b\mathrm{A}^2+2a\mathrm{AB}-3\mathrm{B}^2;\\
&= 9(a^2b^2+18abc-4a^3c-27c^2-4b3)\\
&= 3\{4(a^2-3b)(b^2-3ac)-(ab-9c)^2\}\\
&= \tfrac{1}{3}\{4(a^2-3b)^3-(2a^3-9ab+27c)^2\}.
\end{aligned}$$

If $x=\infty$, we have

$$\mathrm{X}=+\infty, \quad \mathrm{X}_1=+\infty, \quad \mathrm{X}_2=\mathrm{A}.\infty, \quad \mathrm{X}_3=\text{const.},$$

And since, in this case, $m=\pm(3-2i)$, when

$\mathrm{A}>0$	$\mathrm{X}_3>0$	$i=0$	$m=3$,
>0	<0	1	1,
<0	<0	1	1.

These are all the conditions that can obtain, since if $\mathrm{A}<0$, we necessarily have $\mathrm{X}_3<0$, from its last form; it follows also that if $\mathrm{X}_3>0$, A will also be >0; hence the equation has three real roots when

$$\mathrm{X}_3>0, \quad \text{or} \quad a^2b^2+18abc-4a^3c-27c^2-4b^3>0,$$

and but one when

$$\mathrm{X}_3<0, \quad \text{or} \quad a^2b^2+18abc-4a^3c-27c^2-4b^3<0.$$

If $x=0$, the polynomials become

$$\mathrm{X}=c, \qquad \mathrm{X}_1=b, \qquad \mathrm{X}_2=\mathrm{B}, \qquad \mathrm{X}_3=\text{const.}$$

When there is only one real root, its sign is contrary to that of c, but when there are three real roots, or $\mathrm{X}_3>0$ and $i=0$, the number of positive roots is $=m'=k$, the number of changes of signs in these last quantities, and by proceeding as before we find that

if $c>0$, the three roots are all negative, when $b>0$, and $ab>9c$,
2 are positive and 1 negative, in other cases;
if $c<0$, the three roots are all positive, when $b>0$, and $ab<9c$,
1 is positive and 2 are negative, in other cases.

If $a=0$, the properties are reduced to the known ones, that the equation has three real roots when

$$-4b^3-27c^2>0;$$

two of which are positive when $c > 0$,
and only one is positive when $c < 0$;
otherwise, it has only one real root, which has a different sign to that of c.

If $X_2 = 0$, or $a^2b^2 + 18abc - 4a^3c - 27c^2 - 4b^3 = 0$,
the equation has a pair of roots $= \frac{9c - ab}{2a^2 - 6b}$, and the third is $\frac{4ab - a^3 - 9c}{a^2 - 3b}$.

If $B = 0$, or $ab = 9c$, then $X_3 = -b$, and the equation will have three real roots, when $b < 0$, or when a and c have contrary signs; two of which are positive, when $a < 0$, and one positive when $a > 0$.

If $A = 0$, or $a^2 = 3b$, then $X_1 = B = ab - 9c = \frac{1}{3}(a^3 - 27c)$, and the equation has only one real root.

If $A = 0$ and $B = 0$, which can only be when $b = \frac{1}{3}a^2$, and $c = \frac{1}{27}a^3$, the three roots of the equation are each $= -\frac{1}{3}a$.

Δ.

ARTICLE IX.

NOTE, ON A CONTINUED PRODUCT.

BY THE EDITOR.

In equation (5), page 381, Vol. II, if we put

$$k' = -k = \frac{h^2+1}{h},$$

write for r_x its value, and divide the equation by $-P$, we get

$$\Sigma . \frac{1}{k' - 2\cos 2(x\beta + \theta)} = \frac{nh}{1-h^2} . \frac{1-h^{2n}}{1-2h^n\cos 2n\theta + h^{2n}}.$$

Multiply by $\quad dk' = -\frac{1-h^2}{h^2} . dh, \quad$ and integrate,

$$\begin{aligned}
\Sigma . \log\{k' - 2\cos 2(x\beta+\theta)\} &= \int -\frac{ndh}{h} . \frac{1-h^{2n}}{1-2h^n\cos 2n\theta + h^{2n}} \\
&= \int -\frac{d.h^n}{h^n}\left\{1 - \frac{2h^{2n} - 2h^n\cos 2n\theta}{h^{2n} - 2h^n\cos 2n\theta + 1}\right\} \\
&= \int -\frac{d.h^n}{h^n} + \int \frac{d.h^n . (2h^n - 2\cos 2n\theta)}{h^{2n} - 2h^n\cos 2n\theta + 1} \\
&= -\log h^n + \log(h^{2n} - 2h^n\cos 2n\theta + 1),
\end{aligned}$$

and, by restoring the value of k',

$$\Sigma . \log\{h^2 - 2h\cos 2(x\beta+\theta) + 1\} = \log(h^{2n} - 2h^n\cos 2n\theta + 1);$$

or eliminating the logarithms,

$$(h^2 - 2h\cos 2.\overline{\beta+\theta} + 1)(h^2 - 2h\cos 2.\overline{2\beta+\theta} + 1) \ldots . (h^2 - 2h\cos 2.\overline{n\beta+\theta} + 1)$$
$$= h^{2n} - 2h^n\cos 2n\theta + 1.$$

Here $n\beta = i\pi$; and if we take $i = 1$, and put $2n\,\theta = \theta'$, this equation becomes, after transposing the last factor,

$$\left(h^2 - 2h\cos\frac{\theta'}{n} + 1\right)\left(h^2 - 2h\cos\frac{2\pi+\theta'}{n} + 1\right)\ldots\ldots$$
$$\left(h^2 - 2h\cos\frac{2\,(n-1)\,\pi+\theta'}{n} + 1\right)$$
$$= h^{2n} - 2h^n\cos\theta' + 1.$$

This is the well known Theorem of Moivre, from which, had it occurred to me at the proper time, the sum in question (79) might have been deduced. The form in which I have deduced the Theorem, shows that any multiple of π, prime to n, might be used in it, instead of π, as is otherwise obvious.

A form of the theorem of Cotes, immediately deducible from this, is

$$\left(h^2 - 2h\cos\frac{\pi}{2n} + 1\right)\left(h^2 - 2h\cos\frac{3\pi}{2n} + 1\right)\ldots\ldots$$
$$\left(h^2 - 2h\cos\frac{2n-1}{2n}\pi + 1\right)$$
$$= h^{2n} + 1;$$

and from this M. Delaunay deduces a definite integral by a very ingenious process, which may be useful in other cases. It is this:

Each factor in the first member is comprised in the general formula

$$h^2 - 2h\cos\frac{2i+1}{2n}\pi + 1,$$

in which there must be given to i the n successive values

$$0,\ 1,\ 2,\ 3,\ \ldots\ldots\ \overline{n-1};$$

and in order to pass from one factor to the following one, the arc comprised under the sign *cos.* must be increased by the constant quantity $\frac{\pi}{n} = \omega$.

By taking the logarithms, it becomes

$$\Sigma.\log\left(h^2 - 2h\cos\frac{2i+1}{2n}\pi + 1\right) = \log\left(h^{2n}+1\right).$$

Multiply each member by ω, and put $(2i+1)\,\pi = 2nx$: then

$$\Sigma.\,\omega\,\log\left(h^2 - 2h\cos x + 1\right) = \omega\log\left(h^{2n}+1\right)$$
$$= \log\left(h^{2n}+1\right)^{\frac{\pi}{n}};$$

the sign Σ indicating a sum taken relatively to the variable x, which increases from $x = \frac{\pi}{2n}$, to $x = \frac{2n-1}{2n}\pi$, by constant differences, equal to ω.

If, now, we suppose that n becomes infinite, ω will become dx, the sum Σ will be changed into a definite integral taken between the limits $x=0$ and $x=\pi$, and the first member of the equation becomes

$$\int_0^{\pi}\log\left(h^2 - 2h\cos x + 1\right)dx.$$

To determine the second member, two cases must be distinguished:

1°. If $h<1$, $(h^{2n}+1)^{\frac{\pi}{n}}$ is reduced to 1, and we have

$$\int_0^\pi \log(h^2-2h\cos x+1)\,dx=0;$$

2°. If $h>1$, $(h^{2n}+1)^{\frac{\pi}{n}}$ becomes $h^{2\pi}$, and we have

$$\int_0^\pi \log(h^2-2h\cos x+1)\,dx=2\,\pi\log h.$$

M. Poisson finds the same integrals by a very different process in the *Journal de l'Ecole Polytechnique*, Cahier XVII.; and deduces the following, among many others, from them:

Integration by parts gives, in general,

$$\int \log(h^2-2h\cos x+1)\,dx= x\log(h^2-2h\cos x+1)$$

$$-2h\int\frac{x\sin x.dx}{h^2-2h\cos x+1};$$

hence, whence $h<1$,

$$\int_0^\pi \frac{x\sin x.\,dx}{h^2-2h\cos x+1}=\frac{\pi}{h}.\ \log(1+h),$$

and, when $h>1$,

$$\int_0^\pi \frac{x\sin x.\,dx}{h^2-2h\cos x+1}=\frac{\pi}{h}.\ \log\left(1+\frac{1}{h}\right).$$

If $h=1$, these two formulas will be coincident, and give a known result, namely,

$$\int_0^\pi \frac{x\sin x.\,dx}{2\,(1-\cos x)}=\pi.\ \log 2;$$

or, by making $x=2\,z$, and reducing,

$$\int_0^{\frac{1}{2}\pi} z\cot z.\,dz=\tfrac{1}{2}\,\pi.\ \log 2.$$

In these integrals also, h may be imaginary, and if we make

$$h=r\,(\cos\alpha+\sin\alpha\,\sqrt{-1})$$

the first or second, of the several pairs of equations, must be used, accordingly as we have $r<1$, or $r>1$.